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Spectral decomposition of piecewise contracting dynamics on the interval

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Abstract. We study the topological attractors of injective piecewise contracting maps on a compact interval with any finite number $N \geq 2$ of continuity pieces. We prove the existence of a "spectral decomposition" of the attractor into a finite number of transitive components that are either periodic orbits or Cantor sets. In the non-generic case, we prove that some orbits accumulate at both sides of the discontinuities points, and that this phenomenon generates the transitive Cantor sets of the attractor.

Keywords: Interval map, Piecewise contraction, Minimal Cantor sets. **MSC 2010:** 37E05 – 47H09 – 54H20.

Definitions and Spectral Decomposition Theorem

Let $X \subset \mathbb{R}$ be a compact interval. A map $f : X \to X$ is a *piecewise contracting interval map with* $N \geq 2$ *continuity pieces* and *contracting rate* $\lambda \in (0,1)$ (in short (N,λ)-PCIM) if there exists a pairwise disjoint open subintervals collection $X_1, X_2, \ldots, X_N$ such that

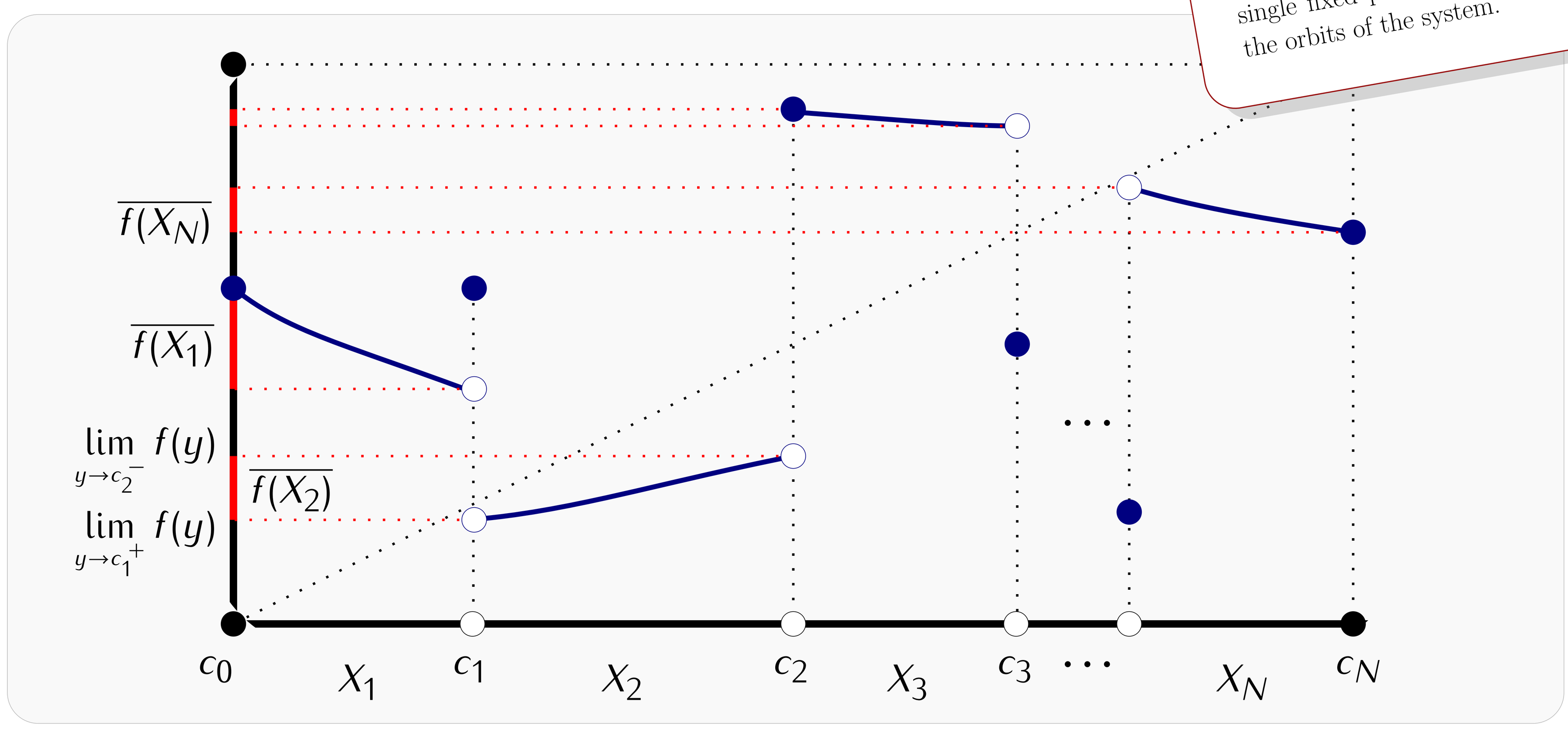
$$X = \bigcup_{1\leq i\leq N} \overline{X_i} \qquad \text{and} \qquad |f(x) - f(y)| \leq \lambda |x-y| \quad \forall\, x, y \in X_i,\ \ \forall\, i \in \{1,2,\ldots,N\}.$$

Also, the map f is supposed to be discontinuous (with unavoidable discontinuities) at the points of the set

$$\Delta := \bigcup_{i\neq j} \partial X_i \cap \partial X_j =: \{c_1, \ldots, c_{N-1}\}.$$

When $N = 1$, the map f is a simple contraction. Thus, there exists a single fixed point that attracts all the orbits of the system.

Graphic of an (N,λ)-PCIM.



The *attractor* of f is defined by the following equalities:

$$\Lambda := \bigcap_{n\geq 1} \Lambda_n \quad \text{where} \quad \Lambda_1 := \overline{f(X\setminus\Delta)} \quad \text{and} \quad \Lambda_{n+1} := \overline{f(\Lambda_n \setminus \Delta)} \quad \forall\, n \geq 1.$$

We say that f satisfies the *separation property* if $f|_{X_i}$ is injective for each i and

$$\overline{f(X_i)} \cap \overline{f(X_j)} = \emptyset \qquad \forall\, i, j \in \{1, \ldots, N-1\} \ :\ i \neq j.$$

Besides, let D be the set of lateral limits of f at their points of discontinuity, i.e.

$$D := \left\{ \lim_{y\to c_i^-} f(y) : 1 \leq i \leq N-1 \right\} \cup \left\{ \lim_{y\to c_i^+} f(y) : 1 \leq i \leq N-1 \right\} \cup \{f(c_0), f(c_N)\}.$$

We are interested in the orbits that do not intersect the set Δ of discontinuities. In this way, we may disregard how f is defined in Δ. Precisely, we construct the set

$$\widetilde{X} := \bigcap_{n\geq 0} f^{-n}(X\setminus\Delta).$$

If f satisfies the separation property, then $\widetilde{X}$ is dense in X.

We say that $A \subset X$ is *pseudo-invariant* if for any $x \in A$ we have

$$\lim_{y\to x^-} f(y) \in A \qquad \text{or} \qquad \lim_{y\to x^+} f(y) \in A.$$

It is easy to prove that sets such as $\omega(x)$ (whenever $x \in \widetilde{X}$) or the attractor Λ are pseudo-invariant.

Note that if $A \subset X$ is pseudo-invariant, then $f(x) \in A$ for any $x \in A \setminus \Delta$ and $A \cap \widetilde{X}$ is invariant. The following is the main result of our work:

Theorem (Spectral Decomposition)

Suppose that f satisfies the separation property and $D \subset \widetilde{X}$, then there exist two natural numbers N_1 and N_2 satisfying $1 \leq N_1 + N_2 \leq 2N$ and such that the attractor Λ of f can be decomposed as follows:

$$\Lambda = \left(\bigcup_{i=1}^{N_1} \mathcal{O}_i\right) \cup \left(\bigcup_{i=1}^{N_2} K_i\right),$$

where $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{N_1} \subset \widetilde{X}$ are pairwise different periodic orbits and $K_1, K_2, \ldots, K_{N_2}$ are transitive pseudo-invariant Cantor sets of X. Moreover, for any $x \in \widetilde{X}$, either there exists $i \in \{1, \ldots, N_1\}$ such that $\omega(x) = \mathcal{O}_i$ or there exists $i \in \{1, \ldots, N_2\}$ such that $\omega(x) = K_i$.

If f is piecewise increasing, then $1 \leq N_1 + N_2 \leq N$.

Short state of art

With respect to periodic attractors, if $\Lambda \cap \Delta = \emptyset$ is known that the asymptotic dynamics is supported by a finite number of periodic orbits (this is also valid in higher dimensions – see [2]). Furthermore, it is known that this behavior is Lebesgue-generic and that generically $N_1 \leq N$ (first proved in the injective case [4, 5] and later in the general case [6], always in one dimension).

With respect to non-generic attractors is known that, for the particular case $N = 2$, the ω-limit set of all point is either a periodic orbit or a Cantor set (see [3]).

Our work generalizes in a certain sense these previous results (we use some complexity tools for this type of systems. The complexity of the dynamics has been also studied in [1]).

Orbits classification and route of the proof

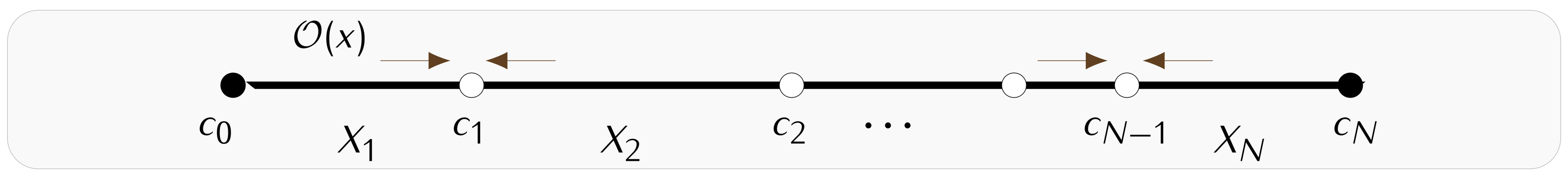
Finally, we say that $c_i \in \Delta$ is *left-right recurrently visited* by the orbit of $x \in \widetilde{X}$, if there exists two strictly increasing sequences $\{t_j\}_{j\geq 0}$ and $\{s_j\}_{j\geq 0}$ of natural numbers such that

$$f^{t_j}(x) \in X_i \ \text{ and } \ f^{s_j}(x) \in X_{i+1} \quad \forall\, j \geq 0, \quad \text{and} \quad c_i = \lim_{j\to\infty} f^{t_j}(x) = \lim_{j\to\infty} f^{s_j}(x).$$

We denote by $\Delta_{lr}(x) \subset \Delta$ the set of discontinuity points that are lr-recurrently visited by the orbit of x, and define

$$\Delta_{lr} := \bigcup_{x\in\widetilde{X}} \Delta_{lr}(x).$$

Graphic situation of the affirmation: $c_1, c_{N-1} \in \Delta_{lr}(x)$.



Suppose that $f : X \to X$ is a (N,λ)-PCIM that satisfies the separation property and that $D \subset \widetilde{X}$.

Lemma 1

The attractor Λ of f can be decomposed as follows:

$$\Lambda = \bigcup_{d\in D} \omega(d). \tag{1}$$

Theorem 2 (periodic ω-limits)

Let $x \in \widetilde{X}$ be such that $\Delta_{lr}(x) = \emptyset$, then $\omega(x)$ is a periodic orbit contained in $\widetilde{X}$.

Theorem 3 (Cantor ω-limits)

Let $x \in \widetilde{X}$ be such that $\Delta_{lr}(x) \neq \emptyset$, then $\omega(x)$ is a Cantor set and $\omega(x) = \overline{\mathcal{O}(y)}$ for all $y \in \omega(x) \cap \widetilde{X}$.

Spectral Decomposition Theorem proof. Applying Theorems 2 and 3 to the points of D, we rewrite (1) as follows:

$$\Lambda = \bigcup_{d\in D} \omega(d) = \left(\bigcup_{i=1}^{N_1} \mathcal{O}_i\right) \cup \left(\bigcup_{i=1}^{N_2} K_i\right), \tag{2}$$

where $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{N_1} \subset \widetilde{X}$ are pairwise different periodic orbits, $K_1, K_2, \ldots, K_{N_2}$ are transitive pseudo-invariant Cantor sets, and $N_1 + N_2 \leq \#D = 2N$. Besides, each periodic orbit $\mathcal{O}_i$, and each Cantor sets K_i, is the ω-limit set of some point $d \in D$.

Now, let us prove that the ω-limit set of any other point in $\widetilde{X}$ also coincides, either with one periodic orbit $\mathcal{O}_i$, or with one Cantor set K_i. First, recall that the ω-limit set $\omega(x)$ of any point $x \in \widetilde{X}$ satisfies $\omega(x) \cap \widetilde{X} \neq \emptyset$. Then, there exists $y \in \omega(x) \cap \widetilde{X}$. Since $\omega(x) \subset \Lambda$, from Lemma 1 we deduce that there exists $d \in D$ such that $y \in \omega(d)$, so $y \in \omega(x) \cap \omega(d) \cap \widetilde{X}$. Besides, $x, d \in \widetilde{X}$, so we can apply Theorems 2 and 3 to deduce that both $\omega(x)$ and $\omega(d)$ are transitive sets. Therefore,

$$\overline{\mathcal{O}(y)} = \omega(x) = \omega(d).$$

This proves that $\omega(x)$ coincides with some set of the decomposition (2). We conclude that, for any $x \in \widetilde{X}$, either there exists $i \in \{1, \ldots, N_1\}$ such that $\omega(x) = \mathcal{O}_i$, or there exists $i \in \{1, \ldots, N_2\}$ such that $\omega(x) = K_i$, ending the proof. ∎

References

[1] Catsigeras, E., Guiraud, P. and Meyroneinc, A. (2018). Complexity of injective piecewise contracting interval maps. *Ergod. Th. & Dyn. Sys.*, 1-25.

[2] Catsigeras, E., Guiraud, P., Meyroneinc, A. and Ugalde, E. (2016). On the asymptotic properties of piecewise contracting maps. *Dynamical Systems*, **31**(2) 107-135.

[3] Gambaudo, J.M. and Tresser, C. (1988). On the dynamics of quasi-contractions. *Bulletin of the Brazilian Mathematical Society*, **19**(1) 61-114.

[4] Nogueira, A. and Pires, B. (2015). Dynamics of piecewise contractions of the interval. *Ergod. Th. & Dyn. Sys.*, **35**(7) 2198-2215.

[5] Nogueira, A., Pires, B. and Rosales, R.A. (2014). Asymptotically periodic piecewise contractions of the interval. *Nonlinearity*, **27**(7) 1603-1610.

[6] Nogueira, A., Pires, B. and Rosales, R.A. (2016). Topological dynamics of piecewise λ-affine maps. *Ergod. Th. & Dyn. Sys.*, 1-18.

Acknowledgments

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