Genericity of Periodic Attractors in Inhibitory Neurons Networks

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Abstract

We study theoretically the dynamical properties of networks composed by a large number of inhibitory neurons, evolving deterministically in real time. We consider the first return map F to a Poincaré section of the phase space and prove that it is piecewise continuous, locally contractive and has the "separation property": different continuity pieces have disjoint images. Then we study the topological dynamics of any abstract discontinuous map under those hypothesis, in a real finite dimensional space. We prove that, generically in the C^0 topology, such systems exhibit one and at most a finite number of persistent periodic sinks attracting all the orbits. We conclude that the neural inhibitory network exhibits C^0-generically a periodic behavior, with a finite number of limit cycles that persist under small perturbations of its structure and thus, under small changes of the idealized model itself.

1 Introduction

We are inspired in a mathematical idealized model of inhibitory neural deterministic networks. In Section 2 we prove that the dynamical system defining this model, in a compact subset of $\mathbb{R}^n$ and evolving with real time t,

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defines a Poincaré map F with discontinuities, that is uniformly contractive in each of its continuity pieces, for some well defined adapted metric. Also we prove that F is locally injective and exhibits what we call the "separation property": different continuity pieces have disjoint images. Therefore it is globally injective.

The results obtained in Section 2 lead to the abstract study of discrete dynamical systems defined by iterations of piecewise continuous and locally contractive maps in a real finite dimensional compact space. In particular, we consider such systems verifying the separation property, and prove in Section 4 that, generically in the C^0 topology, they exhibit one and at most a finite number of persistent periodic sinks attracting all the orbits.

The results in Sections 3 and 4 of this paper are applicable to a very wide class of non linear deterministic discontinuous systems, which are mostly unknown in large (finite) dimensions. For instance they are applicable to the study of the global dynamics of a network of oscillators with negative mutual interactions (phase inhibitions). In fact, an abstract neural inhibitory network can be seen as a system of such interacting oscillators.

We analyze the dynamics of the systems described above in an abstract and theoretical context, using the tools of the Topological Dynamical Systems Theory. The arguments and style of the definitions and proofs, mainly in Sections 3 and 4, are classical in the Dynamical Systems Theory of topological finite-dimensional manifols, in pure (rather than applied) Mathematics.

Recently Bruin and Deane [2] have proved that the contractive piecewise continuous affinities in the plane exhibit periodic behavior, for Lebesgue almost every value of a finite number of real parameters. In any dimension $n \geq 2$, Céssac [7] has proved a similar theorem, for affine piecewise maps modeling a discrete neural network.

As a generalization, the topological dynamical systems that we study in this paper are not affine. As in [2] and [7], our systems have discontinuities along a set that disconnects the n-dimensional metric space into a finite number of continuities pieces, whose interiors are pairwise disjoint. The piecewise continuity and the local contractiveness lead us to obtain the thesis of persistent periodicity in Lemmas 4.2 and 4.3 of this paper.

On the other hand, the separation property will play a fundamental role to obtain the thesis of C^0- density, and thus genericity, of the periodic behavior in the Theorem 4.1. We remark that the separation property is also obtained as a thesis, for a completely connected inhibitory neural network, as a consequence of the model that we analyze in Section 2.

Nevertheless, the hypothesis of separation is not necessary, under other assumptions, to obtain periodicity of an abstract piecewise continuous and locally contractive map. For instance, if each continuous piece is an affinity, the separation property is not needed [2, 7]. We conjecture that the separation property is also unnecessary, if the system is not piecewise-affine but is piecewise C^2 and thus exhibit, in each of its continuity pieces, a property of bounded distortion in the derivative of any future iterate. (Affine maps, in particular, have zero distortion in the derivative).

In Section 2 we develop the idealized and abstract model of the neural deterministic network, from its concrete physical properties, which are classically assumed in Neuroscience, including the case of integrate and fire neurons of pacemaker type [12], and relaxation oscillators.

The network is composed with $n \geq 3$ neurons, where n is arbitrarily large. The neurons are reciprocally coupled by inhibitory synapsis. The internal variable $V_i = V_i(t)$ describing each neuron's potential, for $i = 1, 2 \ldots, n$ evolves increasingly on time t during the interspike intervals: $V_i(t)$ has positive first derivative $dV_i(t)/dt = \gamma_i(V_i(t)) > 0$ and negative second derivative $d^2V_i(t)/dt^2 < 0$, being the solution of a deterministic autonomous differential equation $\dot{x} = \gamma_i(x)$ of a wide general type. We do not assume any numerical values to the parameters of the differential equations, nor any particular formulae to the real functions $\gamma_i(V)$. We study all of the them in a global abstract and qualitative theory.

When the potential V_i reaches a given threshold value, the neuron i produces a spike, and its potential V_i is reset to zero. There are supposed some idealized conditions: all the neurons are inhibitory, there are no delays in the spikes, and the network is totally connected.

When the neuron i spikes, not only its potential V_i changes (being reset to zero), but also (through the synaptical connections) an action potential makes the other $n-1$ neurons suddenly change their respective states. Each spike of the neuron i produces a jump of negative (inhibitory effect) amplitude $-H_{ij} < 0$ in the potential V_j of the neuron $j \neq i$.

The instants of spiking are defined by the evolution of the network itself, and not predetermined by regular intervals of observation of the system. We analyze the state of the system immediately after each spike, in the sequence of instants defined in the network by its own dynamics. The state of the system, after each spike, is a function F of the state after the prior spike. This function F is the so called Poincaré map, that we compute precisely its dependence on the given differential equations and the threshold levels.

The iteration of the Poincaré map is not an artificial discretization of the real time dynamics of the network. On the contrary, the dynamics of F and its properties (for instance periodicity, chaotic attractors) are equivalent to those of the system evolving in real time.

In [9] the technique of the first return Poincaré map to a section transversal to the flux was first applied to study neuron networks. In that paper, an homogeneous network of excitatory coupled pacemakers neurons was analyzed. In [3] the same technique is applied to networks of inhibitory cells, analyzing the real time dynamics via a discrete Poincaré map F. This map is locally contractive and piecewise continuous in a compact set of $\mathbb{R}^{n-1}$. We include the proof of these properties in the section 2 of this paper.

In Section 4, we prove Theorem 4.1, which is the main abstract mathematical result:

Locally contractive piecewise continuous maps with the separation property generically exhibit only periodic asymptotic behavior, with up to a finite number of periodic sinks that are persistent under small perturbations of the map. Generic systems have a topological meaning in this paper: they include a C^0 open and dense family of piecewise continuous systems. The parameters space is not a real space of finite dimension, but the functional space of all the piecewise continuous systems that are locally contractive and have the separation property.

As a consequence we obtain the following applied result:

Generic neuron networks composed by $n \geq 3$ inhibitory cells exhibit only periodic behavior with a finite number of limit cycles that are persistent under small perturbations of the set of parameter values.

This is a result generalizing the conclusions obtained for two neurons networks in [4] and [5]. The persistence of the limit cycles implies a strong result: they persist under small changes of the structure of the system, i.e. of the model itself.

On the other hand, non generic dynamics are structurable unstable: they are destroyed if the system is perturbed, even if the perturbation is arbitrarily small. We refer to those as bifurcating systems, and are out of the scope of this paper.

Even being the mathematical analysis in Sections 3 and 4 of this paper, theoretical and abstract, we observe that its conclusions applied to the dynamics of the model of neuron networks described in Section 2, fit with those obtained by experiments in computer simulations with mutually coupled identical neurons in networks of up to 10^{10} cells, as reported in the following papers:

In [10] it was observed the transition among different periodic activity, indicating that the simulation data fit to experimental and clinical observations. In the computer simulated experiments the alterations of the discharge patterns, when passing from one periodic cycle to another, arise from changes of the network parameters, changes in the connectivity between cells, and also of external modulation.

In [11] the computer simulated experiment shows the dynamics of the network of a large number of coupled neurons. It was observed to be significantly different from the original dynamics of the individual cells: the system can be driven through different synchronization states.

Our thesis of Theorem 4.1 is only applicable to deterministic systems. Nevertheless their conclusions also qualitatively fit with computer simulations of neural systems with randomness [8], [13], which also show the generation of detectable preferred firing sequences.

2 A mathematical model of the inhibitory neurons network.

We include the detailed proof of the mathematical translation from a physical model of n inhibitory pacemaker neurons network, to the dynamics of iterations of a wide class of piecewise continuous contractive maps $F : B \mapsto B$, locally contractive and with the separation property, as first posed in [3], and later in [6]. The model is applicable for any finite number $n \geq 3$ of neurons in the inhibitory network.

The phase space of the system is the compact cube $Q = [-1, 1]^n \subset \mathbb{R}^n$. A point in the phase space is $V = (V_1, V_2, \ldots, V_n)$, describing the potential V_i of each of the neurons $i \in \{1, 2, \ldots, n\}$. We assume that the phase space is normalized: the threshold level of each of the neurons potentials is 1, the maximum of V_i. Also the minimum V_i is normalized to -1, and the reset value of V_i, after a spike of the neuron i, is 0.

Definition 2.1 The physical model. The point V in the phase space Q evolves on time t, during the inter-spike intervals of time, according to an autonomous differential equation and changes without delay in a discontinuous fashion in the exact spiking instants, according to a reseting-synaptical rule. The two regimes, during the interspike interval, and in the spiking instants respectively, are precisely defined according to the following assumptions:

2.1.1 Inter-spike regime. $V_i(t)$ is the solution of a differential equation

$$\frac{dV_i}{dt} = \gamma_i(V_i), \quad \gamma_i : [-1, 1] \mapsto \mathbb{R}, \quad \gamma_i \in \mathcal{C}^1,$$

$$\gamma_i(V_i) > 0, \quad \gamma_i'(V_i) < 0 \ \ \forall\, V_i \in [-1, 1]. \tag{1}$$

where $\mathcal{C}^1$ denotes the space of real functions in $[-1, 1]$, continuous and derivable with continuous derivative in $[-1, 1]$.

The assumption $\gamma_i > 0$ reflects that each neuron potential in the interspike interval is strictly increasing while it does not receive interactions from the other neurons of the network. This comes from the hypothesis that each isolated neuron i is of pacemaker type, i.e. from any initial state $V_i(0) \in [-1, 1)$, the potential spontaneously reaches the threshold level 1 for some time $t = t_i > 0$, if no inhibitory synapsis is received in the time interval $[0, t_i]$.

The assumption $\gamma_i' < 0$, which we call the *dissipative hypothesis,* reflects that the cynetic energy $E_c = (1/2)(dV_i/dt)^2$ is decreasing on time while the spacial vectorial variable $V_i(t)$ freely evolves during the interspike intervals. In fact: $dE_c/dt = (dV_i/dt)(d^2V_i/dt^2) = \gamma_i(V_i)\gamma_i'(V_i)\gamma_i(V_i) < 0$.

The most used example of this type of inter-spike evolution is the relaxation oscillator model of a pacemaker neuron, for which $\gamma_i(V_i) = -\alpha_i V_i + \beta_i$ where $0 < \alpha_i < \beta_i$ are constants. For this type of cells the differential equation (1) is linear, and its solution can be explicitly written:

$$V_i(t) = (\beta_i/\alpha_i) - [(\beta_i/\alpha_i) - V_i(0)]exp(-\alpha_i t).$$

Nevertheless, we won't restrict to that example nor to any other example in the analysis of this paper, but consider the abstract general case given by equation (1).

We define the flux

$$\Phi^t(V) = (\Phi_1^t(V_1), , \Phi_2^t(V_2) \ldots, \Phi_n^t(V_n))$$

as the solution with initial state $V = (V_1, V_2, \ldots, V_n)$ of the differential equations system given by (1). Precisely:

$$\frac{d(\Phi_i^t(V_i))}{dt} = \gamma_i(\Phi_i^t(V_i)) \quad \forall t, \quad \Phi_i^0(V_i) = V_i \tag{2}$$

As $\gamma_i \in \mathcal{C}^1$ we can apply the general theory of differential equations to deduce the following results, as a consequence of the assumptions in (1):

- Two different orbits by the flux do not intersect.
- If B and A are two $(n-1)$-dimensional topological and connected submanifolds of $\mathbb{R}^n$ transversal to the vector field $\gamma = (\gamma_1, \ldots, \gamma_n)$, then the flux transforms homeomorphically any set of initial states in B onto its image set of final states in A.
- For each constant time t it holds the Louville formula:
$$\frac{d(\Phi_i^t(V_i))}{dV_i} = exp \int_0^t \gamma_i'(\Phi_i^s(V_i))\, ds. \tag{3}$$

2.1.3 Spiking-synaptical regime. For each initial state $V \in Q$ the first spiking instant $\bar{t}(V)$ in the network is defined as the first positive time such that at least one of the neurons of the network reaches the threshold level 1. This means that

$$\bar{t}(V) = \min_{1 \leq i \leq n} t_i(V_i), \quad \text{where } \Phi_i^t(V_i) = 1 \Leftrightarrow \ t = t_i(V_i) \tag{4}$$

$$J(V) = \{i \in \{1, 2, \ldots, n\} : \bar{t}(V) = t_i(V_i)\} \tag{5}$$

is the set of neurons that reach the threshold level simultaneously at the instant $\bar{t}(V)$. It is standard to prove that for an open and dense set of initial states (also a full Lebesgue measure set) there is a single neuron i reaching the threshold level first, i.e. $\#J(V) = 1, \;\; J(V) = \{i\}$.

In the spiking instant $\bar{t}$ the reseting and inhibitory synaptical interaction without delay produces an instantaneous discontinuity

$$\sigma : \Phi^{\bar{t}(V)}(V) \mapsto \sigma(\Phi^{\bar{t}(V)}(V))$$

in the state of the system, according to the following formulae:

- If $\#J(V) = 1, \;\; \{i\} = J(V)$ then $\Phi_i^{\bar{t}(V)}(V_i) = 1$, $\Phi_j^{\bar{t}(V)}(V_j) < 1, \; \forall\, j \neq i$, and:

$$\begin{aligned} \sigma^i &= (\sigma_1^i, \sigma_2^i, \ldots, \sigma_n^i), \quad \text{defining:} \\ \sigma_i^i(\Phi^{\bar{t}(V)}(V)) &= 0 \text{ (spiking-reseting rule)} \qquad (6) \\ \sigma_j^i(\Phi^{\bar{t}(V)}(V)) &= \max\{-1, \; \Phi_j^{\bar{t}(V)}(V_j) - H_{ij}\} \;\; \forall\, j \neq i \\ &\quad \text{(synaptic rule)} \qquad (7) \end{aligned}$$

  where $H_{ij} > 0$ is constant, depending only on i, j, and gives the amplitude of the instantaneous negative discontinuity jump $-H_{ij}$ in the potential $\Phi_j^{\bar{t}(V)}(V_j)$ of the neuron $j \neq i$, produced through the inhibitory synaptical connection from neuron i to neuron j.

- If $\#J(V) = k \geq 2, \;\; \{i_1, i_2, \ldots, i_k\} = J(V)$ then σ is multiply defined, having k possible vectorial values $\sigma^{i_1}, \sigma^{i_2}, \ldots, \sigma^{i_k}$, where σ^{i_h} is defined according to formulae (6) and (7).

We also assume that the network, whose nodes are the cells and whose sides are the synaptical inhibitory interactions H_{ij}, is a complete bidirectionally connected graph. Precisely:

$$0 < \epsilon_0 = \min_{i \neq j} H_{ij} \qquad (8)$$

2.1.5 Relatively large dissipation. We assume the following relations between the functional parameters γ_i in the differential equations (1) governing the dissipative interspike regime, and the real parameters $H_{i,j}$ in the formula (7) governing the spiking-synaptical regime.

$$\begin{aligned} \max_{i \neq j} H_{ij} &< \frac{1}{4} \qquad (9) \\ \max_{i,j} \; |\gamma_i(3/4) - \gamma_j(3/4)| &< \frac{\min_i \min_{V_i \in [1/4, 3/4]} |\gamma_i'(V_i)|}{4} \qquad (10) \\ \frac{\max_{i \neq j} H_{ij}}{\min_{i \neq j} H_{ij}} - 1 &< \frac{\min_i \min_{V_i \in [1/4, 3/4]} |\gamma_i'(V_i)|}{4 \, \max_i \; \gamma_i(3/4)} \qquad (11) \end{aligned}$$

Condition (9) assumes that the discontinuity synaptical jumps H_{ij} are not relatively as large as the widest range $[0,1]$ of the potential of the cells when they act as oscillators between the reset value 0 and the threshold level 1, free of synpatical interactions.

The hypothesis (10) and (11) verify for instance for homogeneous networks in which all the functions γ_i and all the synaptic interactions H_{ij} are constant independent of the neurons i, j. But as they are open conditions, they also verify if the network is not homogeneous but the neurons and the synaptical jumps are not very different. Finally they also verify for networks that are very heterogeneous, but the dissipative parameter of the system $\min_i \min_{V_i \in [1/4, 3/4]} |\gamma_i'(V_i)|$ is large enough.

The assumptions above can be also possed for some number $0 < a < 1/2$ instead of $1/4$ in the inequality (9), and $2/(1-2a)$ instead of the denominator 4, in the inequalities (10), (11) . Nevertheless, and without loss of generality, in the computations of this work we will take the assumptions above with $a = 1/4$ for simplicity of the numerical bounds.

2.2 Comments about the physical model.

The hypothesis (9), (10) and (11) will allow us to prove the so called separation property in Theorem 2.9 in this paper. This property will be used to prove the density of the family of systems which exhibit a limit set formed only by a finite number of limit cycles. This denseness leads to the topological genericity of such systems. The key difficult step is due to the non linearities of the system: the Poincaré map, that will be defined in the subsection 2.3.3, is not piecewise affine.

We observe that the assumptions in (1), (6) and (7) are more general that what they a priori seem. In fact, if instead of the variables V_i which describe the electric potentials of each of the neurons, we used other equivalent variables, the vector field γ of the differential equation (1), and the synaptical vectorial interaction σ given by (6) and (7), would have other coordinate expressions.

For instance, each isolated cell i acts as an oscillator, whose potential V_i varies in the interval $[0,1]$. We can diffeomorphically change the variable V_i to a new one $\widehat{V}_i \in [0,1]$, called the *phase* of the oscillator, which by definition, evolves *linearly* with the time t, during a time constant τ_i. In the new variables the differential equation governing the phase state $\widehat{V}_i$ will be $d\widehat{V}_i/dt = 1/\tau_i$ and the flux will be linear in Q.

In [3] it is developed the model in such phase variables $\widehat{V}_i$ for which the flux is linear, and the synaptical inhibitory interactions $-s_{i,j} < 0$ depend on the phase state $\widehat{V}_j$. In a widest model the functions $s_{i,j}(\widehat{V}_j)$ are continuous but not necessarily differentiable.

In resume, up to a change of variables, the model assumed in this paper in hypothesis (1), (6) and (7), includes for instance the model in [3] in which the flux is linear during the interspike interval regime, and the synaptic jumps in the spiking instants adequately depend of the phase V_j of the postsynaptic neuron.

Definition 2.3 The Mathematical Model. In this subsection we will define a Poincaré section $B \subset Q$ of the dynamical system modeling physically the network of n inhibitory neurons defined in 2.1. We shall define the first return Poincaré map $F : B \mapsto B$. We will prove that this map is piecewise continuous, locally contractive and has the separation property. These properties justify the Definition 2.14, at the end of this section, in which we will model and analyze this kind of inhibitory neuron networks through the abstract mathematical discrete dynamical system defined by the iterates of its Poincaré map F.

2.3.1. The Poincaré section B. Let $B \subset Q = [-1,1]^n$ be the compact $(n-1)$-dimensional set defined as follows:

$$B = \bigcup_{k=1}^{n} \widehat{B}_k \quad \text{where} \quad \widehat{B}_k = \{V \in Q : V_k = 0\} \tag{12}$$

The topology in B is defined in each $\widehat{B}_k$ as the induced by its inclusion in the $(n-1)$ dimensional subspace $\{V_k = 0\}$ of $\mathbb{R}^n$. Each $\widehat{B}_k$ is transversal to the flux defined in 2.1.2 solution of the system of differential equations (1), because the vector field γ in the second term of this differential equations has all its components strictly positive.

After each spike, the state of the system is in B, due to the reset rule in equality (6). So the system returns infinitely many times to B from any initial state $V \in Q$.

2.3.2. The partition in continuity pieces. Recalling the definition of the spiking instant $\overline{t}(V)$ in equalities (4), and the definition of the set $J(V)$ of all the neurons that reach the threshold level at time $\overline{t}(V)$, in equality (5), we define the following subset B_i of the Poincaré section B, for any $i = 1, 2 \ldots, n$:

$$B_i = \{V \in B : \ i \in J(V)\} = \{V \in B : \ \overline{t}(V) = t_i(V_i)\} \tag{13}$$

In other words, the set B_i is formed by all the initial states V in the Poincaré section B such that the neuron i reaches the threshold level before or at the same instant than all the other neurons of the network, from the initial state V.

From the implicit equation at right of formulae (4), we deduce that B_i is compact, and that its interior $int(B_i)$ is formed by all the initial states for which $t_i(V_i) < t_j(V_j)$ for all $j \neq i$. Then $int(B_i) \bigcap int(B_j) = \emptyset \ \ \forall \, i \neq j$.

As the flux is strictly increasing inside Q, from any initial state $V \in B$ there exists a finite time $\overline{t}(V)$ defined by equalities (4). Therefore $V \in B_i$

for some not necessarily unique $i \in \{1, 2, \ldots, n\}$. Then $B = \bigcup_{i=1}^n B_i$ and the family of subsets $\{B_i\}_{i=1}^n$ is a topological finite partition of B (i.e. it is a covering of B with a finite number of compact sets whose interiors are pairwise disjoint.)

The compact sets B_i are called *continuity pieces.*

We define the *separation line* S of the partition $\{B_i\}_{i=1}^n$, or *line of discontinuities,* as the union of the topological frontiers ∂B_i of its subsets B_i. Precisely:

$$S = \bigcup_{i=1}^{n} \partial B_i = \bigcup_{i \neq j} (B_i \cap B_j) = B \setminus \left(\bigcup_{i=1}^{n} int B_i \right) \tag{14}$$

2.3.3. The first return Poincaré map F.

The first return map $F : B \mapsto B$ to the Poincaré section $B = \bigcup_{i=1}^n B_i$ is the finite collection of maps $f_i : B_i \mapsto B$ defined as

$$f_i(V) = \sigma^i(\Phi^{\bar{t}(V)}(V)) \;\; \forall\, V \in B_i$$

where Φ is the solution flux defined in 2.1.2 of the system of differential equations (1), $\bar{t}(V)$ is the spiking instant defined by equalities (4) and σ^i is the synaptic vectorial map defined in 2.1.4.

Remark: For simplicity we denote $F|_{B_i} = f_i$ and, when it is previously clear that $V \in B_i$, we simply denote F to refer to the uniquely well defined map f_i.

We observe that F is uniquely defined in $\bigcup_{i=1}^n intB_i$, and multi-defined in the separation line S.

Applying the formulae (4), (6) and (7), we deduce:

$$\begin{aligned}
F|_{B_i}(V) = f_i(V) &= ((f_i)_1, (f_i)_2, \ldots, (f_i)_n) \;\; \forall\, V \in B_i \text{ where} \\
(F|_{B_i})_i(V) = (f_i)_i(V) &= 0 = \max\{-1,\ \Phi_i^{t_i(V_i)}(V_i) - H_{ii}\} \\
(F|_{B_i})_j(V) = (f_i)_j(V) &= \max\{-1,\ \Phi_j^{t_i(V_i)}(V_j) - H_{ij}\} \;\; \forall\, j
\end{aligned} \tag{15}$$

where by convenience we agree to define $H_{ii} = +1$, recalling that $\Phi_i^{t_i(V_i)}(V_i) = +1$.

The formula (15) implies that $f_i = F|_{B_i} : B_i \mapsto B$ is continuous, and, as B_i is compact, then $f_i(B_i)$ is also compact.

The formula (15) changes when one passes from B_i to B_h with $i \neq h$, so F is multidefined in the points of $S = \bigcup_{i \neq h}(B_i \cap B_h)$. Besides F may be discontinuous in $V^0 \in B_i \cap B_h$ because $\lim_{V \in intB_i, V \to V^0} F(V) = f_i(V^0)$ is not necessarily equal to $\lim_{V \in intB_h, V \to V^0} F(V) = f_h(V^0)$.

Remark 2.3.4: We agree to define the image set $F(V)$ of a point $V \in B$ as $\{f_i(V) : i \text{ such that } V \in B_i\}$. The image set $F(V)$ is a single point if $V \in int(B_i)$ because $intB_i$ does not intersect B_j for $j \neq i$. The image set $F(A)$ of a set $A \subset B$ is by definition $F(A) = \bigcup_{V \in A} F(V)$.

We define the *positive reduced Poincaré section* subset $B^+ \subset B$ as:

$$B^+ = \{V \in B : \ 0 \leq V_i \leq 1 - \epsilon_0 \ \ \forall\, i = 1, 2, \dots, n\} \tag{16}$$

where $\epsilon_0 > 0$ is the minimum of the absolute values of the synaptic interactions H_{ij} for $i \neq j$, as assumed in 2.1.4, equality (8). Also, by hypothesis (9) we have

$$0 < \epsilon_0 < \frac{1}{4} \quad , \qquad\qquad \frac{3}{4} < 1 - \epsilon_0 < 1 \tag{17}$$

The dynamics properties of F restricted to the positive Poincaré section B^+, (which will justify the restriction to B^+), will be stated and proved in Theorems 2.4, 2.7, 2.9 and 2.11.

We define $B_i^+ = B^+ \cap B_i$, where B^+ is the positive reduced Poincaré section defined in Equality (16), and B_i are the continuity pieces of the Poincaré map F, defined in Equality (13).

Theorem 2.4 . The return map to the positive Poincaré section
The positive reduced Poincaré section $B^+ \subset B$ defined in (16)*, is forward invariant by the Poincaré map $F|_{B^+} : B^+ \mapsto B^+$, and it is reached from any initial state in B. Even more,*
$F(B^+) \subset \ \bigcup_{i=1}^n \{V \in B : V_i = 0, \quad 0 < V_j \leq 1 - \epsilon_0 \ \ \forall\, j \neq i\} \ \subset B^+$*, and there exists $p \geq 1$ such that $F^p(B) \subset B^+$.*

(Recall that the constant $\epsilon_0 > 0$ defined in Equality (8) verifies the hypothesis (17).)

To prove Theorem 2.4 we will use the following lemma:

Lemma 2.5 *There exists a constant positive minimum time T*

$$T = \frac{\epsilon_0}{\max_k \ \gamma_k(3/4)} > 0$$

such that, if $V \in B$ verifies $V_i \leq 1 - \epsilon_0 \ \ \forall 1 \leq i \leq n$, then the interspike interval $\overline{t}(V) \geq T$.

Proof: According to the formula (4): $\overline{t}(V) = \min_i t_i(V_i)$ where $t = t_i(V_i)$ is the solution of the implicit equation $\Phi_i^t(V_i) = 1$. We integrate the differential equation (1) with initial condition V_i, and recall that $\gamma_i(V_i) > 0$, while the real solution $\Phi_i^s(V_i) \leq 1$ is strictly increasing with s (for V_i constant) and it is the solution of an autonomous differential equation. Using the hypothesis $V_i \leq 1 - \epsilon_0$, and applying the inequality (17), we obtain:

$$\Phi_i^t(V_i) = V_i + \int_0^t \frac{d\Phi_i^s(V_i)}{ds}\, ds = V_i + \int_0^t \gamma_i(\Phi_i^s(V_i))\, ds \tag{18}$$

$$1 = \Phi_i^{t_i(V_i)} = \Phi_i^{t_i(V_i) - \underline{t_i}(V_i)} \left(\Phi^{\underline{t_i}(V_i)}(V_i) \right) = \Phi_i^{t_i(V_i) - \underline{t_i}(V_i)}\,(1 - \epsilon_0)$$

$$1 = 1 - \epsilon_0 + \int_0^{t_i(V_i) - \underline{t_i}(V_i)} \gamma_i(\Phi_i^s(1-\epsilon_0))\, ds$$

where $0 \leq \underline{t_i}(V_i) < t_i(V_i)$ and $\Phi^{\underline{t_i}(V_i)}(V_i) = 1 - \epsilon_0$, being $\underline{t_i}(V_i)$ the time that takes the flux $\Phi_i^t(V_i)$ to be equal to $1 - \epsilon_0$ from the initial state $V_i \leq 1 - \epsilon_0$.

Recall that $\gamma_i(V_i) > 0$ is strictly decreasing with V_i:

$$\gamma_i(\Phi_i^s(1-\epsilon_0)) \leq \gamma_i(\Phi_i^0(1-\epsilon_0)) = \gamma_i(1-\epsilon_0) < \gamma_i(3/4) \quad \forall s \geq 0$$

$$1 \leq 1 - \epsilon_0 + \int_0^{t_i(V_i) - \underline{t_i}(V_i)} \gamma_i(3/4)\, ds$$

$$\epsilon_0 \leq \gamma_i(3/4) \;\; [t_i(V_i) - \underline{t_i}(V_i)] \leq \gamma_i(3/4)\; t_i(V_i)$$

$$\Rightarrow \quad t_i(V_i) \geq T = \frac{\epsilon_0}{\max_k \; \gamma_k(3/4)} \quad \forall\, i, \quad \Rightarrow \quad \overline{t}(V) = \min_i t_i(V_i) \geq T. \;\; \square$$

Proof of Theorem 2.4: It is enough to prove the following two assertions:

Assertion 2.4.A: $F_j(V) \leq 1 - \epsilon_0 \quad \forall\, V \in B$ (even if $V \notin B^+$), $\forall\, j = 1, 2 \ldots, n$.

Assertion 2.4.B: There exists a constant $\epsilon_1 > 0$ such that for all $V \in B_i$, if $F_j(V) \leq 0$ for some $j \neq i$, then $F_j(V) - V_j \geq \epsilon_1$.

Note that the assertion 2.4.B states its thesis in particular if $V \notin B^+$, and also if $V \in B^+$ and $V_j = 0$.

Recall that from the formulae (15) of the Poincaré map F: $F_i(V) = 0$ for all $V \in B_i$. Observe that, being $V_j \geq -1$ for all $V \in B$, from the Assertion 2.4.B we deduce that the first number $p \geq 1$ of iterates of F such that $F^p(B) \subset B^+$ is at most equal to $1 +$ Integer-Part$(1/\epsilon_1)$.

To prove the Assertion 2.4.A, apply the formulae (15) of the return Poincaré map F, and recall the assumptions (8), (9). If $V \in B_i$ then

$$F_i(V) = 0, \quad F_j(V) = \max\{-1, \;\; \Phi_j^{\overline{t}(V)}(V_j) - H_{ij}\} \leq 1 - \min_{i \neq j} H_{ij} = 1 - \epsilon_0 \tag{19}$$

To prove the Assertion 2.4.B, fix $V \in B_i$ such that, for some $j \neq i$

$$F_j(V) \leq 0 \tag{20}$$

Use the formulae (19). We assert that

$$\Phi_j^{\overline{t}(V)}(V_j) < \frac{1}{4} \tag{21}$$

In fact, if it were equal or larger than $1/4$, as $H_{ij} < 1/4$ due to hypothesis (9), the formulae (19) would imply that $F_j(V_j) > 0$ contradicting our hypothesis (20).

Due to the hypothesis of the differential equation (1), the function $\gamma_j(V_j)$ is strictly decreasing with V_j, and the flux Φ_j^t is strictly increasing with t. Use the integrate expression (18) of the differential equation, to compute $\Phi_j^{\bar{t}(V)}(V_j)$, the inequality (21) and the Lemma 2.5, to deduce:

$$0 \leq s \leq \bar{t}(V) \;\;\Rightarrow\;\; V_j \leq \Phi_j^s(V_j) \leq \Phi_j^{\bar{t}(V)}(V_j) < \frac{1}{4} \;\Rightarrow\; \gamma_j(\Phi_j^s(V_j)) > \gamma_j(1/4)$$

$$\Rightarrow \int_0^{\bar{t}(V)} \gamma_j(\Phi_j^s(V_j))\, ds > \gamma_j(1/4) \cdot \bar{t}(V) \geq \min_k \gamma_k(1/4) \cdot T = \frac{\min_k \gamma_k(1/4)\, \epsilon_0}{\max_k \gamma_k(3/4)}$$

Recalling the integral equation (18) and the formula (19) of the return map F, we deduce:

$$F_j(V) - V_j \geq \Phi^{\bar{t}(V)}(V_j) - V_j - H_{ij} = \int_0^{\bar{t}(V)} \gamma_j(\Phi_j^s(V_j))j\, ds - H_{ij}$$

$$F_j(V) - V_j \geq \epsilon_0 \left(\frac{\min_k \gamma_k(1/4)}{\max_k \gamma_k(3/4)} - \frac{\max_{i \neq j} H_{ij}}{\epsilon_0} \right) = \epsilon_1$$

To end the proof it is enough to show that $\epsilon_1 > 0$. Recall from equality (8) that $\epsilon_0 = \min_{i \neq j} H_{ij} > 0$

$$\frac{\epsilon_1}{\epsilon_0} = \frac{\min_k \gamma_k(1/4)}{\max_k \gamma_k(3/4)} - \frac{\max_{i \neq j} H_{ij}}{\min_{i \neq j} H_{ij}} =$$

$$= \frac{\min_k \gamma_k(1/4) - \max_k \gamma_k(3/4)}{\max_k \gamma_k(3/4)} - \left(\frac{\max_{i \neq j} H_{ij}}{\min_{i \neq j} H_{ij}} - 1 \right)$$

$$\frac{\epsilon_1}{\epsilon_0} = \frac{\gamma_h(1/4) - \gamma_h(3/4) + \gamma_h(3/4) - \gamma_k(3/4)}{\max_k \gamma_k(3/4)} - \left(\frac{\max_{i \neq j} H_{ij}}{\min_{i \neq j} H_{ij}} - 1 \right)$$

where we have taken h and k such that $\gamma_h(1/4) = \min_k \gamma_k(1/4), \quad \gamma_k(3/4) = \max_k \gamma_k(3/4)$.

$$\frac{\epsilon_1}{\epsilon_0} \geq \frac{\gamma_h(1/4) - \gamma_h(3/4) - |\gamma_h(3/4) - \gamma_k(3/4)|}{\max_k \gamma_k(3/4)} - \left(\frac{\max_{i \neq j} H_{ij}}{\min_{i \neq j} H_{ij}} - 1 \right)$$

$$\frac{\epsilon_1}{\epsilon_0} \geq \frac{\gamma_h(1/4) - \gamma_h(3/4) - \max_{h \neq k} |\gamma_h(3/4) - \gamma_k(3/4)|}{\max_k \gamma_k(3/4)} - \left(\frac{\max_{i \neq j} H_{ij}}{\min_{i \neq j} H_{ij}} - 1 \right)$$

Applying the mean value theorem of the derivative of γ_h, which is negative due to the dissipation hypothesis of the differential equation in assumption (1), we obtain

$$\gamma_h(1/4) - \gamma_h(3/4) = \left(\frac{3}{4} - \frac{1}{4} \right) \cdot (-\gamma'_h(\chi))\big|_{\chi \in [1/4, 3/4]} \geq$$

$$\geq \frac{\min_i \, \min_{V_i \in [1/4, 3/4]} |\gamma'_i(V_i)|}{2}$$

The last inequalities and the assumptions (10) and (11) of relative large dissipativity, imply:

$$\frac{\epsilon_1}{\epsilon_0} > \frac{\min_i \min_{V_i \in [1/4, 3/4]} |\gamma_i'(V_i)|}{\max_k \gamma_k(3/4)} \left(\frac{1}{2} - \frac{1}{4} - \frac{1}{4}\right) = 0 \ \Box$$

Remark 2.6 Formula of the Poincaré map in B^+.

As a consequence of Theorem 2.4, from now on we will restrict the Poincaré map F to the positive section B^+. In fact, from the statements of Theorem 2.4 it is deduced that the forward dynamics and the limit set of the orbits to the future, of the restricted F, will be the same as those of F in the whole Poincaré section B.

Due to Theorem 2.4 if $V \in B^+$ then $F(V) \subset B^+$. Therefore $(F|_{B_i^+})_j(V) \geq 0 \ \ \forall \, i, j$. Using the formula (15) we can rewrite the expression of the Poincaré map:

$$\begin{aligned}
F|_{B_i^+}(V) = f_i(V) &= ((f_i)_1, (f_i)_2, \ldots, (f_i)_n) \ \ \forall \, V \in B_i^+ \text{ where} \\
(F|_{B_i^+})_i(V) = (f_i)_i(V) &= 0 \ = \ \Phi_i^{t_i(V_i)}(V_i) - H_{ii} \\
(F|_{B_i^+})_j(V) = (f_i)_j(V) &= \Phi_j^{t_i(V_i)}(V_j) - H_{ij} \ \ \forall \, j
\end{aligned} \tag{22}$$

Theorem 2.7 Local injectiveness of the Poincaré map.

The Poincaré map F defined in formulae (22), restricted to each of its positive continuity pieces B_i^+ defined in 2.3.7, *is injective.*

Proof: Fix a continuity piece B_i^+ of F in the positive Poincaré section B^+. The piece B_i^+ will remain fixed along this proof. Therefore we will denote F instead of f_i.

Take $V, W \in B_i^+$.such that $F(V) = F(W) \in B^+$. We must prove that $V = W$.

Due to the formulas (22) of the Poincaré map: $F_i(V) = F_i(W) = 0$ and

$$F_j(V) = \Phi_j^{\bar{t}(V)}(V_j) + H_{ij} = \Phi_j^{\bar{t}(W)}(W_j) + H_{ij} = F_j(W) \quad \forall \, j$$

As H_{ij} is constant, we deduce that

$$\Phi_j^{\bar{t}(V)}(V_j) = \Phi_j^{\bar{t}(W)}(W_j) \quad \forall \, j$$

Therefore the vectorial flux $\Phi^t(V) \in Q = [-1, 1]^n$ defines an orbit from the initial $V \in B^+$ that intersects the orbit from the initial state $W \in B^+$. Two different orbits of the flux do not intersect. Then, the two orbits are the same. If necessary changing the roles of V and W, we deduce that

$$\Phi^{t_0}(V) = W \text{ for some } t_0 \geq 0.$$

Recall that $B^+ \subset B$, so V has at least one component $V_k = 0$, and none of them is negative (recall (12) and (16)). But Φ^t_j is the strictly increasing in time solution of the differential equation $d\Phi^t_j/dt = \gamma_j(\Phi^t_j)$, with $\gamma_j > 0$ for all j.

We deduce that if $V \in B^+$, and if $\Phi^{t_0}(V) = W$ for $t_0 > 0$, then $W_j > 0 \; \forall j$, and therefore $W \notin B^+$. As we know that $W \in B^+$ and $t_0 \geq 0$, we conclude that $t_0 = 0$, and then $W = \Phi^{t_0}(V) = \Phi^0(V) = V$. $\square$

Definition 2.8 The Separation Property. We say that F verifies *the separation property* if

$$f_i(B_i^+) \cap f_j(B_j^+) = \emptyset \quad \forall i \neq j$$

where $\{B_i^+\}$, $i = 1, 2, \ldots, n$, are the continuity pieces of F in the positive Poincaré section B^+, as defined in 2.3.7., and f_i is the continuous expression of $F|_{B_i^+}$ according to the formulae (22).

Note that B_i^+ is compact for all i, and F is continuous in each B_i^+. Therefore the image $F(B_i^+)$ is a compact set. Then, the separation property implies that *there exists a minimum positive distance $\alpha > 0$ between the images by F of two different continuities pieces.*

Theorem 2.9 *The Poincaré map F verifies the separation property.*

Proof: Take B_i^+ and B_j^+ with $i \neq j$. The formulae (22) of the Poincaré map $F|_{B^+} : B^+ \subset B \mapsto B$ and the Theorem 2.4 imply that

$$\forall\, V \in B_i^+ : \;\; (f_i)_i(V) = 0, \quad (f_i)_j(V) > 0 \;\; \forall\, j \neq i$$

$$\forall\, W \in B_j^+ : \;\; (f_j)_j(V) = 0, \quad (f_j)_i(V) > 0 \;\; \forall\, i \neq j$$

Then $f_i(B_i^+) \bigcap f_j(B_j^+) = \emptyset$. $\square$

Remark 2.10 Global injectiveness of the Poincaré map.

From Theorems 2.7 and 2.9 it is deduced that the Poincaré map F is globally injective in B^+. In fact, if $V \neq W$ are in the same continuity piece B_i^+, then $f_i(V) \neq f_i(W)$ because $F|_{B_i^+} = f_i$ is injective. And if $V \neq W$ respectively belong to two different continuity pieces B_i^+ and B_j^+ for $i \neq j$, then $f_i(V) \neq f_j(W)$ because $f_i(B_i^+) \cap f_j(B_j^+) = \emptyset$, due to the separation property. We deduce that if $V \neq W$ then $F(V) \bigcap F(W) = \emptyset$, where the image set $F(V)$ of a point is defined in the Remark 2.3.4.

Theorem 2.11 Local contractiveness.

The Poincaré map $F|_{B_i^+}$ is uniformly contractive, but not infinitely contractive, in each of its continuity pieces B_i^+. Precisely, there exist two constant real numbers $0 < \sigma < \lambda < 1$ and a distance dist in the positive Poincaré section $B^+ = \bigcup B_i^+$, such that, for all $i = 1, 2, \ldots, n$:

$$\sigma\, dist(V, W) \leq dist(f_i(V), f_i(W)) \leq \lambda\, dist(V, W) \quad \forall\, V, W \in B_i^+$$

where $f_i : B_i^+ \mapsto B^+$ is the continuous restriction of F to B_i^+, according with formulae (22).

Remark: The distance dist of Theorem 2.11 induces the same topology in B^+ as a subset of $\mathbb{R}^{n-1} \subset \mathbb{R}^n$. In fact, along the proof of the Theorem 2.11 we will construct a linear projection $\pi : \mathbb{R}^n \mapsto \mathbb{R}^{n-1}$ and a diffeomorphism $\xi : \mathbb{R}^n \mapsto \mathbb{R}^n$ of C^1 class, such that:

$$\mathrm{dist}(V, V + dV) = \|\pi(d\xi dV)\| \text{ where } \|\cdot\| \text{ is a norm in } \mathbb{R}^n.$$

Proof of the Theorem 2.11:

The continuity piece $B_i^+$is fixed. For simplicity of the notation, along this proof we will use simply F to denote f_i.

The existence of the distance dist and the contraction rate λ is proved in the Theorem 3 of [6]. For a seek of completeness we include here some pieces of the proof of [6], adding to them the existence of the lower bound contraction rate $0 < \sigma < 1$, $\sigma < \lambda$.

Due to the Tubular Flux Theorem there exists a $\mathcal{C}^1$ diffeomorphism which is a spatial change of variables $\xi : V \mapsto \breve{V}$ from $Q \subset \mathbb{R}^n$ onto $\breve{Q} \subset \mathbb{R}^n$, such that $\xi|_{B^+} = id$ and the solutions of the differential equation (1) in Q verify

$$d\breve{V}/dt = \vec{a}$$

in $\breve{Q}$, where $\vec{a} \in \mathbb{R}^n$ is a constant vector with positive components. It verifies:

$$\xi(\phi^t(V)) = \xi(V) + \vec{a} \cdot t, \;\; d\xi \cdot \gamma(V) = \vec{a} \;\; \forall\, V \in Q$$

Define in $\mathbb{R}^n$ the ortogonal projection π onto the $(n-1)$-dimensional subspace

$$a_1\breve{V}_1 + a_2\breve{V}_2 + \ldots + a_n\breve{V}_n = 0$$

The flux of the differential equation (1), after the change ξ of variables in the space, is ortogonal to that subspace, and is transversal to $\xi(B^+) = \breve{B}^+ = B^+$ (recall that $\xi|_{B^+}$ is the identity map).

Consider any real function $g : \mathbb{R}^{(n-1)} \mapsto \mathbb{R}$:

$$\forall\, \breve{V}, \breve{V} + d\breve{V} \in \mathbb{R}^n : \qquad \pi(d\breve{V}) = \pi(d\breve{V} + g(\breve{V}) \cdot \vec{a}).$$

$$\forall\, V,\; V + dV,\;\; U \in \breve{B}_k,\; \text{define} \quad \mathrm{dist}(V, V + dV) = \|\pi(d\xi\, dV)\|$$

$$\mathrm{dist}(V, U) = \int_0^1 \|\pi(d\xi_{V+t(U-V)} \cdot (U - V)\|\, dt \qquad (23)$$

It is left to prove that $f_i : B_i^+ \mapsto B^+$ is contractive with this distance.

Let us apply f_i to V and $V + dV$ in $\in B_i^+$. We use the equalities (22).

We shall use the Liouville derivation formula of the flux of the differential equation respect to its initial state, with constant real time t:

$$d\Phi_j^t/dV_j = \exp\left(\int_0^t \gamma_j' \left(\Phi_j^s(V_j)\right) ds\right)$$

Define:

$$-\alpha = \max_j \max_{V_j \in [-1,1]} \gamma_j'(V_j) \quad < 0, \qquad -\alpha^* = \min_j \min_{V_j \in [-1,1]} \gamma_j'(V_j) \quad < 0$$

Use the Lemma 2.5 to bound uniformly above zero the inter-spike intervals $\overline{t}(V)$:

$$0 < T \leq \overline{t}(V)$$

Recall that $\overline{t}(V)$ is the solution of the C^1 implicit equation $\Phi^{\overline{t}(V_i)}(V_i) = 1$. Then $\overline{t}(V)$ is a continuous real function of $V \in B^+$, and B^+ is a compact set. So, $\overline{t}(V)$ is also upper bounded by a constant:

$$\overline{t}(V) \leq T^*$$

Derive the formulae (22) to obtain:

$$F(V + dV) - F(V) = dF \cdot dV = [(\partial F_j/\partial V_j)dV_j + (\partial F_j/\partial V_i)dV_i]_{1 \leq j \leq n}$$

$$\partial F_j/\partial V_j \; = \; (d\Phi_j^t(V_j)/dV_j)\big|_{t=\overline{t}(V)} =$$

$$= \exp\left(\int_0^{\overline{t}(V)} \gamma_j'(\Phi_j^s(V_j))\, ds\right) \in [e^{-\alpha^* T^*}, e^{-\alpha T}] \tag{24}$$

$$\partial F_j/\partial V_i \; = \; (d\Phi_j^t(V_j)/dt)\big|_{t=\overline{t}(V)} \cdot (dt_i(V_i)/dV_i) g(V) \cdot \gamma_j(\Phi_j^{\overline{t}(V)}(V_j))$$

where $g(V) = dt_i(V_i)/dV_i$ is the real function obtained deriving respect to V_i the implicit equation given in (4): $1 = \Phi_i^{t_i(V_i)}(V_i)$. Call $\vec{e}_j$ to the $j-$th. vector of the canonic base in $\mathbb{R}^n$ and join all the results above:

$$\begin{gathered}
\pi \cdot d\xi \; (F(V + dV) - F(V)) = \pi \cdot d\xi \cdot dF \cdot dV = \\
= \pi \cdot d\xi \left(\textstyle\sum_{j=1}^n (\partial F_j/\partial V_j) \cdot dV_j \vec{e}_j\right) + \pi \cdot d\xi (g(V) \cdot \gamma(\Phi^{\overline{t}(V)}(V)) = \\
= \pi \cdot d\xi \left(\textstyle\sum_{j=1}^n (\partial F_j/\partial V_j) \cdot dV_j\, \vec{e}_j\right) + g(V) \cdot \pi \cdot d\xi \cdot \gamma(\Phi^{\overline{t}(V)}(V))) = \\
= \pi \cdot d\xi \left(\textstyle\sum_{j=1}^n (\partial F_j/\partial V_j) \cdot dV_j\, \vec{e}_j\right) + g(V) \cdot \pi(\vec{a}) = \\
= \pi \cdot d\xi \left(\textstyle\sum_{j=1}^n (\partial F_j/\partial V_j) \cdot dV_j\, \vec{e}_j\right)
\end{gathered} \tag{25}$$

We define the numbers σ and λ: $0 < \sigma = e^{-\alpha^* T^*} < e^{-\alpha T} = \lambda < 1$ and observe from the computations in (24) that:

$$0 < \sigma = e^{-\alpha^* T^*} \leq \partial F_j/\partial V_j \leq e^{-\alpha T} = \lambda < 1$$

Applying the definition of the differential distance in (23), and the equality (25), we obtain:

$$\begin{aligned}\operatorname{dist}(F(V), F(V+dV)) &= \|\pi(d\xi \cdot dF \cdot dV)\| \leq \\ &\leq \lambda \, \|\pi(d\xi \cdot dV)\| = \lambda \operatorname{dist}(V, V+dV) \\ \operatorname{dist}(F(V), F(V+dV)) &= \|\pi(d\xi \cdot dF \cdot dV)\| \geq \\ &\geq \sigma \, \|\pi(d\xi \cdot dV)\| = \sigma \operatorname{dist}(V, V+dV)\end{aligned}$$

By integration of the formula (23) we conclude:

$$\sigma \operatorname{dist}(V, U) \leq \operatorname{dist}(F(V), F(U)) \leq \lambda \operatorname{dist}(V, U) \;\; \square$$

Remark 2.12 Local homeomorphic property of the Poincaré map.
Each continuity piece f_i of the Poincaré map in B_i^+ is an homeomorphism onto its image.

It is an immediate consequence of Theorem 2.11 and the global injectiveness of F: the continuous restriction $f_i = F|_{B_i^+}$, is Lipschitz with constant $\lambda < 1$ and its inverse (defined from $f_i(B_i) \mapsto B_i$) is also Lipschitz with constant $1/\sigma > 1$. Then f_i is an *homeomorphism* onto its image. $\square$

In the following corollary we resume all the conclusions of this section:

Corollary 2.13 *If the network of n inhibitory neurons verifies the assumptions of the physical model,* evolving with real time t in the phase space $Q \subset \mathbb{R}^n$ as stated in (1), (6), (7), (8), (9), (10) and (11), *then there exists a Poincaré section B^+ and a return map $F : B^+ \mapsto B^+$,* with the following properties:

a) *F is piecewise continuous.* Precisely: there exists a finite partition $\{B_i^+\}_{1 \leq i \leq n}$ of the Poincaré section B^+, formed by compact sets $B_i^+ \subset B^+$ with pairwise disjoint interiors, and there exist n continuous maps $f_i : B_i^+ \mapsto B^+$, being $F(V) = \{f_i(V) : \; i \text{ such that } V \in B_i^+\}$ for all $V \in B^+$. As a consequence F is uniquely defined as f_i in the interior of its continuity piece B_i, and multi-defined as f_i, f_j in $B_i \cap B_j$, if $i \neq j$.

b) *F is locally uniformly contractive and not infinitely contractive,* i.e. for some metric dist in B^+ the exist constants $0 < \sigma < \lambda < 1$ such that for all $1 \leq i \leq n$: $\sigma \operatorname{dist}(V, W) \leq \operatorname{dist}(f_i(V), f_i(W)) \leq \lambda \operatorname{dist}(V, W) \;\; \forall \, V, W \in B_i^+$.

c) *F has the separation property,* i.e. $f_i(B_i^+) \cap f_j(B_j^+) = \emptyset$ if $i \neq j$. Therefore, there exists $0 < \alpha = \min_{i \neq j} \operatorname{dist}(f_i(B_i^+), f_j(B_j^+))$.

Note that from b) and c), it is deduced that F is globally injective in B^+, as proved in Remark 2.10. Also from b) it is deduced that $f_i : B_i^+ \mapsto f_i(B_i^+) \subset B^+$ is an homeomorphism onto its image, as proved in the remark 2.12.

Due to Corollary 2.13, all the general results that we will prove for abstract piecewise continuous maps F verifying a), b), c), are applicable to the networks of inhibitory neurons in the assumptions of the physical model stated in 2.1. Nevertheless the reciprocal of the Corollary 2.13 does not hold. Given a map F verifying a), b), c) there does not necessarily exist a network of inhibitory neurons in the hypothesis of the physical model stated in 2.1 for which F is its first return Poincaré map.

We wide the scenario of possible models of inhibitory neuronal networks. In fact, the properties a), b) c) are open (in the uniform $\mathcal{C}^0$ + *Lipschitz* topology of the finite family of maps f_i). Thus they are not only verified by systems for which the differential equations (1) are independent in the n variables V_i, but also if the system is of the form $dV/dt = \gamma^*(V)$, where $\gamma^* : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a $\mathcal{C}^1$ vector field, near enough the given $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$, even if γ^* does not verify all the hypothesis stated in the Definition (2.1).

Also the matrix $(H_{i,j})_{i,j}$ of synaptic interactions in the network can be substituted for any matrix $(H^*_{i,j})_{i,j}(V)$, not necessarily constant, but functions near the constant matrix $(H_{i,j})_{i,j}$ and so, still verifying the assumptions (8), (9), (10), (11). Therefore, without changing the synaptical rules in equations (6) and (7), but allowing the synaptic interactions slightly depend of the postsynaptic potentials, we will obtain a Poincaré map F still verifying the thesis a), b), c) of the Corollary 2.13.

Besides, as observed in the subsection 2.2, the physical model includes looser hypothesis than those specified in 2.1, modulus any differentiable change of the variables of the system. So, also in those models the properties a), b), c) are verified by an open family of systems.

Finally, the properties a) b) c) of the Corollary 2.13 are verified by many other models, in which the interspike regime is stated as a dynamical system depending continuously on time t and on the initial state V, but not necessarily as regular as to verify a differential equation. The dynamics of the potential V_i in the inter-spike interval may be given by a flux $\Phi_i^t(V_i)$ defined continuously in time t, strictly increasing on t, continuous but not necessarily differentiable respect to t nor to the initial state.

The arguments above inspire to wide the abstract mathematical model of a network of n inhibitory neurons, according to the following definition:

Definition 2.14 The Abstract Mathematical Model. We say that a map $F : B^+ \mapsto B^+$, in a set B^+ homeomorphic to a compact ball of $\mathbb{R}^{n-1}$, *models a generalized network of* n *inhibitory neurons* if it verifies the statements a), b), c) of the Corollary 2.13.

3 The abstract dynamical system.

Let $B \subset \mathbb{R}^n$ be a compact set, homeomorphic to a compact ball of $\mathbb{R}^{n-1}$. In particular B_i is connected.

Definition 3.1 A *finite partition* of B is a finite collection $\{B_i\}_{1\leq i\leq m}$ of compact non empty sets $B_i \subset B$, such that $\bigcup_{1\leq i\leq m} B_i = B$ and $\text{int } B_i \cap \text{int } B_j = \emptyset$, for $i \neq j$.

Denote $S = \bigcup_{i\neq j} B_i \cap B_j = \bigcup_{i=1}^m \partial B_i$ and call S the *separation line,* or *line of discontinuities,* (although it is not a line in the usual sense, but the union of the topological frontiers of B_i).

Definition 3.2 Given a finite partition $\{B_i\}_{1\leq i\leq m}$ of B, we call F a *piecewise continuous map on* $(B, \mathcal{P})$ *with the separation property* if F is a finite family $F = \{f_i\}_{1\leq i\leq m}$ of *homeomorphisms* $f_i : B_i \mapsto f_i(B_i) \subset int(B)$, such that $f_i(B_i) \cap f_j(B_j) = \emptyset$ if $i \neq j$. We note that F is multi-defined in the separation line S.

Each B_i shall be called a *continuity piece* of F.

Remark 3.3 A piecewise continuous map F with the separation property is globally injective because it is an homeomorphism in each continuity piece and two different continuities pieces have disjoint images. Therefore F^{-1} exists, uniquely defined in each point of $F(B) = \bigcup_i f_i(B_i)$. In fact:

For any point $x \in \bigcup_i f_i(B_i)$, its backward first iterate is uniquely defined as $F^{-1}(x) = f_i^{-1}(x)$, where i is the unique index value such that $x \in f_i(B_i)$.

Nevertheless F^{-1} is not necessarily injective because F is multidefined in $S = \bigcup_{i\neq j}(B_i \cap B_j)$.

F^{-1} is continuous in $F(B)$, because $F^{-1}|_{f_i(B_i)} = f_i^{-1}$ and f_i is an homeomorphism due to the Definition 3.2.

Definition 3.4 We say that F is uniformly *locally contractive* if there exists a constant $0 < \lambda < 1$, called an *uniform contraction rate for F,* and a metric dist in B, such that $\text{dist}(f_i(x), f_i(y)) \leq \lambda \text{dist}(x, y)$, for all x and y in the same B_i, for all $1 \leq i \leq m$.

Given a point $x \in B$, its image set is $F(x) = \{f_i(x) : x \in B_i\}$. If $H \subset B$, its image set is $F(H) = \bigcup_{x\in H} F(x)$. We have that $B \supset F(B) \supset \ldots F^k(B) \supset \ldots$.

The second iterate of the point $x \in B$ is the set $F^2(x) = F(F(x))$. It is analogously defined the $j-$th. iterate as the set $F^j(x)$ for any $j \geq 1$. We convene to define $F^0(x) = \{x\}$ and $F^0(H) = H$.

Definition 3.5 For any natural number $k \geq 1$, we call *atom of generation* k to

$$f_{i_k} \circ \ldots \circ f_{i_2} \circ f_{i_1}(B_{\mathbb{I}})$$

where $\mathbb{I} = (i_1, i_2, \ldots, i_k) \in \{1, 2, \ldots, m\}^k$ and $B_{\mathbb{I}}$ is the subset of B_{i_1} where the composed function above is defined. (If $B_{\mathbb{I}}$ were an empty set, then the atom is empty.) Abusing of the notation we write the atom as:

$$f_{i_k} \circ \ldots \circ f_{i_2} \circ f_{i_1}(B_{i_1})$$

We note that each atom of generation k is a compact, not necessarily connected set, whose diameter is smaller than $\lambda^k \text{diam} B$.

The set $F^k(B)$ is a compact set, formed by the union of all the not empty atoms of generation k. There are at most m^k and at least m not empty atoms of generation k, where m is the number of continuity pieces of F.

Definition 3.6 Given $x_0 \in B$, a future orbit $o^+(x_0)$ is a sequence of points $\{x_i\}_{i\geq 0}$, starting in x_0, such that $x_{i+1} \in F(x_i) \;\; \forall\, i \geq 0$. Due to the multi-definition of F in the separation line S, the points of S and those that eventually fall in S may have more than one future orbit.

A point y is in the limit set $L^+(o^+(x_0))$ of a future orbit of x_0 if there exists $k_j \to +\infty$ such that $x_{k_j} \to y$.

The limit set $L^+(x_0)$ is the union of the limit sets of all the future orbits of x_0.

The *limit set* $L^+(B)$ of the map F, also denoted as $L^+(F)$, is the union of the limit sets $L^+(x)$ of all the points $x \in B$.

Remark 3.7 Due to the compactness of the space B the limit set $L^+(o^+(x_0))$ of any future orbit, *is not empty.* It is standard to prove that $L^+(o^+(x_0))$ *is compact* (because it is closed in the compact space B). Nevertheless $L^+(x_0)$ may be not compact, if the point x_0 has infinitely many different future orbits. Finally, we assert that $L^+(o^+(x_0))$ *is invariant:* $F^{-1}(\, L^+(o^+(x_0))\,) = L^+(o^+(x_0))$.

Proof: Consider $y \in L^+(o^+(x_0))$. We have $y = \lim_{j\to+\infty} x_{k_j} \in F(B)$ if $k_j \geq 1$.

$F^{-1} : F(B) \to B$ is a continuous uniquely defined function (see Remark 3.3). Then $x_{k_j - 1} = F^{-1}(x_{k_j}) \to F^{-1}(y)$, so $F^{-1}(y) \in L^+(o^+(x_0))$ proving that

$$F^{-1}(\, L^+(o^+(x_0))\,) \subset L^+(o^+(x_0))$$

Let us prove the converse inequality: $F^{-1}(\, L^+(o^+(x_0))\,) \supset L^+(o^+(x_0))$.

$F = \{f_i : B_i \mapsto B\}$ is defined and continuous in each of its finite number of pieces B_i, that are compact sets that cover B. Then there exists some $i \in \{1, 2, \ldots, n\}$ and a subsequence (that we still call k_j), such that

$$y = \lim_{j\to+\infty} x_{k_j} \in B_i, \quad \forall j \geq 0: \;\; x_{k_j} \in B_i, \;\; x_{k_j+1} = f_i(x_{k_j}),$$

$$f_i(y) = \lim f_i(x_{k_j}) = \lim x_{k_j+1}$$

We conclude that there exists $y_1 = f_i(y) \in F(y)$ such that $y_1 \in L^+(o^+(x_0))$. In other words, $y \in F^{-1}(L^+(o^+(x_0))$. This last assertion was proved for any $y \in L^+(o^+(x_0))$. Therefore $L^+(o^+(x_0)) \subset F^{-1}(\, L^+(o^+(x_0))\,)$ as wanted. $\square$

Definition 3.8 We say that a point x is *periodic of period* p if there exists a first natural number $p \geq 1$ such that $x \in F^p(x)$. This is equivalent to x be a periodic point in the usual sense, for the uniquely defined map F^{-1}, i.e. $F^{-p}(x) = x$ for some first natural number $p \geq 1$.

We call the backward orbit of x (i.e. $\{F^{-j}(x), j = 1, \ldots, p\}$), a periodic orbit with period p.

It is not difficult to show that the limit set $L^+(F)$ is contained in the compact, totally disconnected set $K_0 = \bigcap_{k \geq 1} F^k(B)$. It could be a Cantor set. But generically K_0 shall be the union of a finite number of periodic orbits, as we shall prove in Theorem 4.1.

Definition 3.9 We say that F is *finally periodic* with period p if the limit set $L^+(F)$ is the union of *only* a finite number of periodic orbits with minimum common multiple of their periods equal to p. In this case we call *limit cycles* to the periodic orbits of F.

We call basin of attraction of each limit cycle L to the set of points $x \in B$ whose limit set $L^+(x)$ is L.

Topology in the space of piecewise continuous locally contractive maps in B.

Let $\mathcal{P} = \{B_i\}_{1 \leq i \leq m}$ and $\mathcal{Q} = \{A_i\}_{1 \leq i \leq m}$ be finite partitions (see Definition 3.1) of the compact region B with the same number m of pieces.

We define the distance between $\mathcal{P}$ and $\mathcal{Q}$ as

$$d(\mathcal{P}, \mathcal{Q}) = \max_{1 \leq i \leq m} \quad \text{Hdist}(A_i, B_i) \tag{26}$$

where $\text{Hdist}(A_i, B_i)$ denotes the Hausdorff distance between the two compact sets A_i and B_i. i.e.

$$\text{Hdist}(A_i, B_i) = \max\{\text{dist}(x, B_i), \text{dist}(y, A_i) : x \in A_i, y \in B_i\}$$

and $\text{dist}(x, B_i) = \min\{\text{dist}(x, y) : y \in B_i\}$

Although it is standard to check the following properties of the distance between two partitions $\mathcal{P}$ and $\mathcal{Q}$, we include their proofs for a seek of completeness:

Remark 3.10 . If $d(\mathcal{P}, \mathcal{Q}) < \epsilon$ then:

- Hdist $(S, \widehat{S}) < \epsilon$, where $S = \cup_i(\partial B_i)$ is the separation line of the partition $\mathcal{P} = \{B_i : 1 \leq i \leq m\}$, and $\widehat{S} = \cup_i(\partial A_i)$ is the separation line of the partition $\mathcal{Q} = \{A_i : 1 \leq i \leq m\}$.

- For all $i \neq j$ such that $B_i \cap A_j \neq \emptyset$, and for all $p \in B_i \cap A_j$:

$$\text{dist}(p, S) < \epsilon, \;\; \text{dist}(p, \widehat{S}) < \epsilon.$$

Proof: In the following proof we will use that B is homeomorphic to a compact ball in $\mathbb{R}^{n-1}$: it is a compact and connected metric space and so, all the subsets $M \subset B$ have the following property:

$$y \notin M \quad \Rightarrow \quad \text{dist}(y, M) = \text{dist}(y, \partial M)$$

where ∂M is the topological frontier of M as a subset of the topological space B.

To deduce that $\text{Hdist}\,(S, \widehat{S}) < \epsilon$, recall that $S = \cup_{i=1}^{m} \partial B_i$, $\widehat{S} = \cup_{i=1}^{m} \partial A_i$. So, it is enough to prove that $\text{Hdist}(\partial B_i, \partial A_i) < \epsilon$ for all i. If $\partial B_i = \partial A_i$ then their Hausdorff distance is zero and thus, smaller than ϵ. On the other case, there exists $p \in (\partial B_i \setminus \partial A_i) \cup (\partial A_i \setminus \partial B_i)$. First suppose $p \in \partial B_i,\ p \notin \partial A_i$.

$$d(\mathcal{P}, \mathcal{Q}) < \epsilon \ \Rightarrow \ \text{dist}(p, A_i) < \epsilon \ \ \forall\, p \in B_i, \text{ in particular } \ \forall\, p \in \partial B_i$$

If $p \notin \partial A_i$ then $\text{dist}(p, A_i) = \text{dist}(p, \partial A_i)$

$$\Rightarrow \ \text{dist}\ (p, \partial A_i) < \epsilon \quad \forall\, p \in \partial B_i$$

Changing the roles of A_i and B_i, the same argument works for $q \in \partial A_i \setminus \partial B_i$. So we deduce

$$\text{Hdist}\ (\partial A_i, \partial B_i) =$$

$$= \max\{\text{dist}(p, \partial A_i), \text{dist}(q, \partial B_i),\ p \in \partial B_i,\ q \in \partial A_i\} < \epsilon$$

Let us prove now the second assertion in this remark. We will only prove that $\text{dist}(p, \widehat{S}) < \epsilon \ \ \forall\, p \in B_i \cap A_j$. The inequality $\text{dist}(p, S) < \epsilon$ follows from this one, changing the roles of the partitions $\mathcal{P}$ and $\mathcal{Q}$.

If $p \in B_i \cap A_j$ then, being $i \neq j$, we deduce

$$\text{int}A_i \cap \text{int}A_j = \emptyset \ \Rightarrow \ \text{int}A_i \cap A_j = \emptyset \ \ \Rightarrow \ p \notin \ \text{int}A_i \ \ \Rightarrow \ \ p \in B_i \setminus (\text{int}A_i)$$

$$\Rightarrow \ \text{dist}(p, \text{int}A_i) = \text{dist}(p, \partial A_i) \leq \text{Hdist}(B_i, A_i) < \epsilon$$

But $\partial A_i \subset \widehat{S}$, then $\text{dist}(p, \partial A_i) \geq \text{dist}(p, \widehat{S})$. So we deduce $\text{dist}(p, \widehat{S}) < \epsilon$ as wanted. $\square$

Definition 3.11 Let $F = \{f_i : B_i \mapsto B\}_{1 \leq i \leq m}$ and $G = \{g_i : A_i \mapsto B\}_{1 \leq i \leq m}$ be locally contractive piecewise continuous maps on $(B, \mathcal{P})$ and $(B, \mathcal{Q})$ respectively. Given $\epsilon > 0$ we say that G is a *ϵ-perturbation of* F if

$$\max_{1 \leq i \leq m} \left\| (g_i - f_i)|_{B_i \,\cap\, A_i} \right\|_{\mathcal{C}^{\,0}} < \epsilon, \ \ |\lambda_F - \lambda_G| < \epsilon \quad \text{and} \quad d(\mathcal{P}, \mathcal{Q}) < \epsilon$$

where λ_F denotes *the uniform contraction rate* of F in its continuity pieces, defined in 3.4, and $\|\cdot\|_{\mathcal{C}^0}$ denotes the $\mathcal{C}^0$ distance in the functional space of continuous functions defined in a *compact* set K:

$$\|(g - f)|_K\|_{\mathcal{C}^0} = \max_{x \in K} \text{dist}(g(x), f(x))$$

Definition 3.12 We say that the limit cycles of a finally periodic map F (see Definition 3.9) are *persistent* if:

For all $\epsilon^* > 0$ there exists $\epsilon > 0$ such that all ϵ-perturbations G of F are finally periodic with the same finite number of limit cycles (periodic orbits) than F, and such that each limit cycle L_G of G has the same period and is ϵ^*-near of some limit cycle L_F of F (i.e. the Hausdorff distance between L_G and L_F verifies $\text{Hdist}(L_G, L_F) < \epsilon^*$).

Definition 3.13 Denote $\mathcal{S}$ to the *space of all the systems that are piecewise continuous with the separation property and locally contractive,* according with the Definitions 3.2 and 3.4.

We say that a property $\mathbb{P}$ of the systems in $\mathcal{S}$ (for instance being finally periodic as we will show in Theorem 4.1) is *(topologically) generic* if $\mathbb{P}$ is verified, at least, by an *open and dense* subfamily of systems in the functional space $\mathcal{S}$, with the topology in $\mathcal{S}$ defined in 3.11.

Precisely, being *generic* means:

1) The *openness* condition: For each piecewise continuous map F that verifies the property $\mathbb{P}$ there exist $\epsilon > 0$ such that all ϵ-perturbation of F also verifies $\mathbb{P}$.

2) The *denseness* condition: For each piecewise continuous map F that does not verify the property $\mathbb{P}$, given $\epsilon > 0$, arbitrarily small, there exist some ϵ-perturbation G of F such that G verifies the property $\mathbb{P}$.

The openness condition implies that the property $\mathbb{P}$ shall be robust under small perturbations of the system. It is robust under small changes, not only of a finite number of real parameters, but also of the functional parameter that defines the model itself. So the system should be structurally stable. When this robustness holds, the property $\mathbb{P}$ is still observed when the system, the model itself, does not stay exactly fixed, but is changed, even in some unknown fashion, remaining near the original one.

The density condition combined with the openness condition, means that the only behavior that have chance to be observed under not exact experiments are those that verify the property $\mathbb{P}$. In fact, if the system did not exhibit the property $\mathbb{P}$, then some arbitrarily small change of it, would lead it to exhibit $\mathbb{P}$ robustly.

The denseness condition implies that if the property $\mathbb{P}$ were generic, then the opposite property (Non-$\mathbb{P}$) has null interior in the space of $\mathcal{S}$ of systems, i.e. Non-$\mathbb{P}$ is not robust: some arbitrarily small change in the system will lead it to exhibit $\mathbb{P}$. That is why we define the following:

Definition 3.14 If the property $\mathbb{P}$ is generic, we say that any system that does not exhibit $\mathbb{P}$ is *bifurcating,* and Non-$\mathbb{P}$ is a *not persistent* property.

4 The generic persistent periodic behavior.

Theorem 4.1 *Let F be a locally contractive piecewise continuous map with the separation property. Then generically F is finally periodic with persistent limit cycles.*

To prove Theorem 4.1 we shall use the following lemmas 4.2 and 4.3:

Lemma 4.2 *If there exists an integer $k_0 \geq 1$ such that the compact set $K_0 = F^{k_0}(B)$ does not intersect the separation line S of the partition into the continuity pieces of F, then F is finally periodic.*

Proof: By hypothesis $\text{dist}(K_0, S) = d > 0$, because K_0 and S are disjoint compact sets. On the other hand

$$K_0 = F^{k_0}(B) = \bigcup_{A \in \mathcal{A}_{k_0}} A$$

where $\mathcal{A}_k$ for any fixed $k \geq 1$, denotes the family of all the atoms of generation k defined in 3.5.

The diameter diam (A) of each atom A of the finite family $\mathcal{A}_k$, is smaller than $\text{diam}(B)\,\lambda^k$. Therefore it converges to zero when $k \to +\infty$. Thus, for all k large enough:

$$\text{diam}(A) \leq \frac{d}{2} \quad \forall\, A \in \mathcal{A}_k$$

It is not restrictive to suppose $k \geq k_0$. Then $A \subset F^k(B) \subset F^{k_0}(B) = K_0 \quad \forall\, A \in \mathcal{A}_k$.

We assert that each atom $A \in \mathcal{A}_k$ for such k, is contained in the interior of some continuity piece B_i. To prove this last assertion we give the following argument (P), that will be useful also in the proof of Lemma 4.3:

(P) Fix a point $x \in A$. As the continuities pieces cover the space B, there exists some (a priori not necessarily unique) index i such that $x \in B_i$. It is enough to prove that $y \in \mathit{int}(B_i)$ for all $y \in A$ (including x itself).

We argue in the compact and connected metric space B, using the following known properties of the metric space B with the topology induced by its inclusion in $\mathbb{R}^{n-1}$, as a subset homeomorphic to a compact ball.

- The triangular property.

- The distance $\text{dist}(y, M)$ of a point $y \notin M$, to a set $M \subset B$, is the same that the distance of y to the topological frontier ∂M of M as a subset of B.

- $\text{dist}(y, M_1) \geq \text{dist}(y, M)$ if $M_1 \subset M$.

We denote B_i^c to the complement of B_i in B, and in the topology relative to B we denote: $\overline{(B_i^c)}$ to the closure of B_i^c, i.e the complement of $\mathit{int}(B_i)$, and ∂B_i to the frontier of B_i in B, $\partial B_i \subset S$:

$$\text{dist}(x, y) \leq \mathit{diam}(A) < d/2,$$

$$\mathrm{dist}(x, \overline{(B_i^c)}) = \mathrm{dist}(x, \partial B_i) \geq \mathrm{dist}(x, S) \geq d$$

$$\mathrm{dist}(y, \overline{(B_i^c)}) \geq \mathrm{dist}(x, \overline{(B_i^c)}) - \mathrm{dist}(x, y) \geq d - d/2 = d/2 > 0$$

Therefore $y \notin \overline{(B_i^c)}$ proving that $y \in \ int(B_i)$ as wanted. $\Box$ **(P)**

We deduce that given an atom $A \in \mathcal{A}_k$, there exists and is unique a natural number i_0 such that $A \in int(B_{i_0})$. Therefore $F(A)$ is a single atom of generation $k+1$.

From the definition of atom in 3.5, we obtain that any atom of generation larger than k is contained in an atom of generation k. But each atom of generation k is in the interior of a piece of continuity of the partition $\{B_i\}$. We deduce that there exists a sequence of natural numbers $\{i_h\}_{h\geq 0}$, called *the itinerary of the atom A,* such that

$$A \in \ int(B_{i_0}), \ \ F(A) = f_{i_0}(A) \subset \ int(B_{i_1}),$$

$$F^2(A) = f_{i_1} \circ f_{i_0}(A) \subset \ int(B_{i_2}), \ldots \tag{27}$$

and the successive images of the atom A of generation k, are single atoms of generation $k+1, k+2, \ldots, k+h, \ldots$. Therefore, the successive images of the atom A, in the sequence (27), are contained in a sequence of atoms: $A = A_0, A_1, A_2, \ldots, A_h, \ldots$, all of generation k.

The same property holds for any of these atoms of generation k, and each of them is contained in the interior of a continuity piece of F, so F is uniquely defined there and we have:

$$A = A_0 \subset \ int(B_{i_0}), \ \ F(A_0) \subset A_1 \subset \ int(B_{i_1}),$$

$$F^2(A_0) \subset F(A_1) \subset A_2 \subset \ int(B_{i_2}), \ldots, \tag{28}$$

For fixed k, the family of atoms of generation k is finite, so we conclude that there exists two first natural numbers $0 \leq h < h+p$ such that $F^p(A_h) \subset A_h$.

Note that, $F^p(A_h)$ is uniquely defined as $f_{i_{h+p}} \circ f_{i_{h+p-1}} \circ \ldots \circ f_{i_h}$, because we are considering sets contained in the interior of the continuity pieces of F.

Due to the uniform contractiveness of f_i in each of its continuities pieces, $F^p : A_h \mapsto A_h$, is uniformly contractive. The Banach Theorem of the Fixed Point states that in a complete metric space, any uniformly contractive map from a compact set to itself, has an unique fixed point, and all the orbits in the set converge to this fixed point in the future. Therefore, there exists in A_h a periodic point p_0 by F, of period $p \geq 1$, and all the orbits with initial states in A_h have the periodic orbit L of p_0, as their limit set.

By construction A_h contains the image of A by an iterate F^h, uniquely defined. So we conclude that the limit set of all the points in the atom A is L.

The construction above can be done starting with any initial atom $A \in \mathcal{A}_k$. And $\mathcal{A}_k$ is a finite family. We conclude that there exists one, and at most a finite number of periodic limit cycles, attracting all the orbits of $\bigcup_{A\in\mathcal{A}_k} A = F^k(B)$.

The last assertion implies that the limit set of B is formed by that finite family of periodic limit cycles, ending the proof of this lemma. □

Lemma 4.3 *In the hypothesis of Lemma* 4.2, *the limit cycles of F are persistent.*

Proof: We shall prove that the limit cycles are persistent according to the definition 3.12.

The condition of the hypothesis of Lemma 4.2 is open in the topology defined in 3.11, because K_0 and S are compact and at positive distance. Therefore, there exists $\epsilon_0 > 0$ such that, for all $0 < \epsilon < \epsilon_0$, all $\epsilon-$perturbation G of F, is finally periodic.

(Q) We claim that, *given $k_0 \geq 1$ fixed such that dist$(A, S) \geq d > 0$ for all $A \in \mathcal{A}_{k_0}(F)$, then there exists $0 < \epsilon < \epsilon_0$ small enough such that if G is a $\epsilon-$perturbation of F, then there is a one-to-one bijection Ψ between the families $\mathcal{A}_k(F)$ and $\mathcal{A}_k(G)$, of the atoms of all generation $k \geq 1$ of F and G respectively, and besides, for some k large enough, the itinerary of each of the atoms $A \in \mathcal{A}_k(F)$ is the same than the itinerary of the respective atom $\Psi(A) = \widehat{A} \in \mathcal{A}_k(G)$.*

In fact, due to the definition of ϵ_0- perturbation of F, the continuity pieces $B_i = B_i(F) \subset B$ and $\widehat{B}_i = B_i(G) \subset B$, of F and G respectively, are correspondent by a one-to-one bijection, such that the Haussdorff distance

$$\text{Hdist}(B_i(F), B_i(G)) < \epsilon_0.$$

On the other hand, for all $k \geq 1$, the atoms $A \in \mathcal{A}_k(F)$ and $\widehat{A} \in \mathcal{A}_k(G)$, due to the definition of atom in 3.5, are:

$$A = F^k(B_{\mathbb{I}}), \quad \widehat{A} = G^k(B_{\widehat{\mathbb{I}}})$$

identified by words

$$\mathbb{I} = (i_1, i_2, \dots, i_k), \quad \widehat{\mathbb{I}} = (\widehat{i}_1, \widehat{i}_2, \dots, \widehat{i}_k) \quad \in \{1, 2, \dots, m\}^k$$

We define the correspondence

$$\Psi(A) = \widehat{A} \quad \text{if and only if} \quad \widehat{\mathbb{I}} = \mathbb{I}$$

With $k = k_0$ fixed, we have

$$\text{dist }(A, S) \geq d > 0 \quad \forall\, A \in \mathcal{A}_{k_0}(F). \tag{29}$$

On the other hand, due to the definition 3.11 of ϵ_1-perturbation G of F, we have $\|f_i - g_i\|_{C^0} < \epsilon_1$, $\quad$ H $\text{dist}(B_i, \widehat{B}_i) < \epsilon_1 \quad \forall\ i = 1, 2, \ldots, m$. But the finite composition of C^0 diffeomorphisms depends continuously of the diffeomorphisms in the topology defined in 3.11. Then, for k_0-fixed, there exists $0 < \epsilon_1 < \epsilon_0$ such that

$$|f_i - g_i\|_{C^0} < \epsilon_1, \quad \text{H dist}(B_i, \widehat{B}_i) < \epsilon_1 \quad \forall\ i = 1, 2, \ldots, m \quad \Rightarrow$$

$$\|f_{i_{k_0}} \circ f_{i_{k_0 - 1}} \circ \ldots \circ f_{i_1}(B_{i_1}) \ - \ g_{i_{k_0}} \circ g_{i_{k_0 - 1}} \circ \ldots \circ g_{i_1}(\widehat{B}_{i_1})\| < \frac{d}{3}$$

$$\forall\ \mathbb{I} = (i_1, i_2, \ldots, i_{k_0}) \in \{1, 2, \ldots, m\}^{k_0}$$

In other words, the last statement can be reformulated as:

$$G \text{ is a } \epsilon_1 - \text{perturbation of } F \quad \Rightarrow$$

$$\text{Hdist } (A, \Psi(A)) < \frac{d}{3} \quad \forall\ A \in \mathcal{A}_{k_0}(F) \ \text{ and } \ \widehat{A} = \Psi(A) \in \mathcal{A}_{k_0}(G).$$

Besides, if $\epsilon_1 > 0$ is chosen smaller than $d/3$, from the definition 3.11 we obtain

$$\text{H dist}(S, \widehat{S}) < \epsilon_1 < \frac{d}{3}$$

where $\widehat{S}$ is line of discontinuities of the piecewise continuous map G, which is a $\epsilon-$ perturbation of F. Joining the last two inequalities with (29) and applying the triangular inequality, we deduce:

$$\text{dist } (\widehat{A}, \widehat{S}) \geq d - \frac{d}{3} - \frac{d}{3} = \frac{d}{3} \quad \forall\ A \in \mathcal{A}_{k_0}(G)$$

We conclude that if $0 < \epsilon_1 < d/3$, and if G is a ϵ_1-perturbation of F, then the atoms $\widehat{A} \in \mathcal{A}_{k_0}(G)$ remain at distance larger than $d/3 > 0$ from the separation line $\widehat{S}$ of G, and at Hausdorff distance smaller than $\epsilon_1 < d/3$ of its corresponding atom $A \in \mathcal{A}_{k_0}(F)$, being $d = \text{dist}(K_0, S)$ and $K_0 = \cup\{A \in \mathcal{A}_{k_0}(F)\} = F^{k_0}(B)$.

Recall that

$$k \geq k_0 \quad \Rightarrow \quad K = F^k(B) \subset F^{k_0}(B) = K_0, \quad G^k(B) \subset G^{k_0}(B)$$

$$A' \in \mathcal{A}_k(F) \Rightarrow A' \subset A \in \mathcal{A}_{k_0}(F), \ \ \widehat{A}' \in \mathcal{A}_k(G) \Rightarrow \widehat{A}' \subset \widehat{A} \in \mathcal{A}_{k_0}(G)$$

$$\Rightarrow \text{dist}(\widehat{A}', \widehat{S}) \geq \text{dist}(\widehat{A}, \widehat{S}) \geq d = \text{dist}(K_0, S).$$

So, using the same positive numbers $d > 0$ and $0 < \epsilon_1 < d/3$ for all $k \geq k_0$, we obtain the following statement:

(S) There exists $k_0 \geq 1$ and $0 < \epsilon_1 < d/3$, such that if G is a ϵ_1-perturbation of F, then for all $k \geq k_0$ the atoms $\widehat{A} \in \mathcal{A}_k(G)$ remain at distance larger than $d/3 > 0$ from the separation line $\widehat{S}$ of G, and at Haussdorf distance smaller than $\epsilon_1 < d/3$ of its corresponding atom $A \in \mathcal{A}_k(F)$, being $d = \text{dist}(K_0, S)$, $K_0 = \cup\{A \in \mathcal{A}_{k_0}(F)\} = F^{k_0}(B)$.

Now, $k_0 \geq 1$, $\epsilon_1 > 0$ and $d > 0$ are fixed as in statement (S), and the generation $k \geq k_0$ is chosen and also *fixed,* such that the atoms of $\mathcal{A}_k(F)$ and of $\mathcal{A}_k(G)$ have all diameter smaller than $d/6$.

Repeating the argument (P) used in the proof of Lemma 4.2, we deduce that $A \in \mathcal{A}_k(F)$ is in the interior of some (and unique) continuity piece B_{i_k} of F: $A \subset int(B_{i_k})$, and all the points at distance smaller than $d/3$ of A are contained in $int(B_{i_k})$. This last includes the atom $\widehat{A} = \Psi(A) \in \mathcal{A}_k(G)$. Then $\widehat{A} \subset int(B_{i_k})$. Repeating once more the same argument (P) used in the proof of Lemma 4.2, now with G instead of F, we deduce $\widehat{A} \subset int(\widehat{B}_{\widehat{i}_k})$ for some unique $\widehat{i}_k$.

(T) We assert that for the fixed $k \geq k_0$ constructed as above, for any $A \in \mathcal{A}_k(F)$, $\widehat{A} = \Psi(A) \in \mathcal{A}_k(G)$, the indexes i_k and $\widehat{i}_k$ constructed as above, coincide: $\widehat{i}_k = i_k$.

By contradiction, if $\widehat{i}_k \neq i_k$ then, aplying Remark 3.10, the distance from any point $p \in B_{i_k} \cap B_{\widehat{i}_k}$ to $\widehat{S}$ is smaller than Hdist $(S, \widehat{S}) < \epsilon_1$. Then $\text{dist}(A, \widehat{S}) < \epsilon_1 < d/3$ contradicting the statement (S). $\square$ **(T)**

So i_k is the first index of the itinerary of the atom $A \in \mathcal{A}_k(F)$, which due to (T) coincides with the first index $\widehat{i}_k$ of the itinerary of the atom $\widehat{A} = \Psi(A) \in \mathcal{A}_k(G)$. Now let us prove that the indexes of the itinerary of A and of $\widehat{A}$, i.e. the indexes for their future iterates, also coincide.

The future iterate of any atom of generation k, is an atom of generation $k' \geq k$ by F, and also by G. And they are contained in some atoms of generation k of F, and G respectively. Therefore, using (T), the images of an atom $A \in \mathcal{A}_k(F)$ or $\widehat{A} = \Psi(A) \in \mathcal{A}_k(G)$, by all the future iterates of F or of G respectively, are in the interior of their respective one-to-one corresponding continuity pieces $B_{i_{k'}}$, $\widehat{B}_{i_{k'}}$, where the index $i_{k'}$ is the same for all $k' \geq k$. Then the itineraries of A and $\widehat{A} = \Psi(A)$ are the same, as we asserted in (Q). $\square$ **(Q)**

As a consequence of assertion (Q), the indexes $i_0, i_1, i_2, \ldots$ in the finite chain of atoms denoted in (27) and (28), remain unchanged, for F or for G, being G an $\epsilon-$perturbation of F for $\epsilon > 0$ small enough. We deduce the following statement:

A: *The number of periodic orbits in the atoms of generation k, and their periods, remain unchanged, when substituting F by any ϵ-perturbation G, if $\epsilon > 0$ is sufficiently small.*

Now it is standard to prove by induction on $k \geq 1$ the following property:

Let F be a piecewise continuous contractive map with contraction rate upper bounded by $0 < \lambda < 1$. Let $\epsilon > 0$ such that $\lambda + \epsilon = \widehat{\lambda} < 1$. Let G be an $\epsilon-$ perturbation of F. Then, for all $k \geq 1$, each atom $\widehat{A}$ of generation k

for G, is at distance smaller than $\sum_{j=0}^{k-1} 2\epsilon\,\widehat{\lambda}^j < 2\epsilon/(1-\widehat{\lambda}) = \epsilon^* > 0$ of the respective atom A of generation k for F, with the same itinerary than $\widehat{A}$.

Therefore we deduce the following statement:

B: *Any periodic point found in an atom $\widehat{A}$ of generation k for G, is at distance smaller than ϵ^* than the respective periodic point found in the corresponding atom A for F with the same itinerary.*

The statements A and B imply that the limit cycles are persistent according to Definition 3.12. □

Remark 4.4 In the proof of Lemmas 4.2 and 4.3, we did not use the separation property $f_i(B_i) \cap f_j(B_j) = \emptyset \;\; \forall i \neq j$. At the very beginning of the proof of Lemma 4.3, we obtained that the piecewise continuous and locally contractive systems verifying the thesis of the Lemma 4.2, even if they do not have the separation property, contain an open family of systems in the topology defined in 3.11. Then:

In the space of all the piecewise continuous and locally contractive systems (even if they do not have the separation property), those whose limit set is formed by a finite number of persistent limit cycles form an open family.

Nevertheless, to prove the genericity of the periodic persistent behavior, we need to prove that the family of periodic maps is dense in the space of systems. In the proof of the Theorem 4.1, to obtain the density property, we shall restrict to the space of systems $\mathcal{S}$ that verify the separation property.

Remark 4.5 From the proof of Lemma 4.2, the first integer $k_0 \geq 1$ such that $F^{k_0}(B) \bigcap S = \emptyset$ may be very large, and so the period p may be very large.

In fact, if the system has $n \approx 10^{12}$ neurons, and if no neuron becomes dead, i.e. it does not eventually remain forever under the threshold level without giving spikes, then the periodic sequences $i_1, \ldots, i_p$, defined as the itinerary of the periodic limit cycles, have inside the period p, at least once each of all the indexes $i \in \{1, 2, \ldots, n\}$. Then $p \geq n \approx 10^{12}$.

As we have shown in the proof of the Lemma 2.5, there exists a minimum time $T > 0$ between two consequent spikes. Suppose for instance that $T \approx 10\ [ms]$ and $n \approx 10^{12}$. The lasting time of the periodic sequence could be approximately $10^{-3} \times 10^{12}[s] = 10^9[s] \geq 31$ years. So, if most of the neurons did not become dead, the observation of the theoretical periodic behavior of the inhibitory system in the future, could not be practical during a reasonable time of experimentation, and only the irregularities inside the period could be registered, showing the system as virtually chaotic.

Proof of Theorem 4.1. Due to Lemma 4.2 the existence of a finite number of limit cycles attracting all the orbits of the space is verified at least for those systems in the hypothesis of 4.2. This hypothesis is an open condition because $K_0 = F^{k_0}(B)$ and S are compact set at positive distance, and for fix k_0, the set $F^{k_0}(B)$ depends continuously on the map F.

To prove its genericity it is enough to prove that the hypothesis of Lemma 4.2 is also a dense condition in the space $\mathcal{S}$ of piecewise continuous contractive maps with the separation property, with the topology in $\mathcal{S}$ defined in 3.11.

Take F being not finally periodic.

We shall prove that, for all $\epsilon > 0$ there exists a $\epsilon-$ perturbation G of F that verifies the hypothesis of Lemma 4.2, and thus G is finally periodic with persistent limit cycles.

Let be given an arbitrarily small $\epsilon > 0$.

The contractive homeomorphisms f_i of the finite family $F = \{f_i : B_i \mapsto B\}_i$, with contraction rate $0 < \lambda < 1$, can be C^0 extended to

$$F_\epsilon = \{f_{i,\epsilon} : U_i \mapsto B\}_i, \quad \text{where} \quad f_{i,\epsilon} : U_i \mapsto B, \quad f_{i,\epsilon}|_{B_i} = f_i,$$

U_i is a compact neighborhood such that $B_i = \overline{B}_i \subset int(U_i) \subset U_i = \overline{U}_i \subset B$, and $f_{i,\epsilon}$ is an homeomorphism onto its image.

We construct $f_{i,\epsilon}$ still contractive in U_i, with a contraction rate

$$0 < \lambda' < 1 \text{ such that } |\lambda - \lambda'| < \epsilon. \tag{30}$$

Such a finite family F_ϵ of continuous extensions $f_{i,\epsilon}$ to open sets $U_i \supset B_i$, exists as an application of Tietze Theorem (see for instance Theorem 2.15 of [1]), applied to homeomorphisms.

The role of the family F_ϵ of continuous extensions $f_{i,\epsilon}$ will be the following:

The union of the domains of $f_{i,\epsilon}$ is the union of the sets $U_i \subset B$. They do not form a partition of B because they overlap in sets with non void interiors, covering the discontinuity line S of the given F. So F_ϵ is multi-defined now, not only in S but in the set

$$V = \bigcup_{i \neq j} U_i \cap U_j \supset S$$

with non void interior. The covering $\{U_i\}$ makes the line of discontinuities S a kind of fuzzy set: i.e. one can move freely the line of discontinuities S inside the interior of the set V, to define a new partition of the space B.

Our purpose is to find some G that is a ϵ- perturbation of F, such that G verifies the hypothesis of Lemma 4.2. We will choose not any G, but someone in a very particular way, obtained from F moving *only* the line S of discontinuities of F to a new line $S_{\mathcal{Q}} \subset V$, and the partition $\mathcal{P}$ of continuity pieces of F to a near new partition $\mathcal{Q}$. We will do that *without changing the functional values of F in the points where it was already defined.*

The image of B by the future n-th. iterate of F_ϵ, includes the image of B by F^n, because $f_{i,\epsilon}$ is defined in a set $U_i \subset B_i$ (recall that B_i is the domain of f_i), and $f_{1,\epsilon}|_{B_i} = f_i$.

But the image of B by the future $n-$th. iterate of F_ϵ, includes also the image of B by G^n (being G any piecewise contractive function G that is a restriction of F_ϵ to some continuity pieces $C_i \subset U_i$). Then, $F_\epsilon^n(B)$ includes the image of B by the iterate of all those ϵ- perturbation G of F, obtained from F moving only its line of discontinuities, and so, changing only the partition $\mathcal{P} = \{B_i\}$ of the continuity pieces to a new partition $\mathcal{Q} = \{C_i\}$ such that $C_i \subset U_i$ (without changing the functional values of F in $B_i \cap C_i$).

In other words, the extended family F_ϵ is the "egg" of all the $\epsilon-$ perturbations G of F, obtained from F moving *only* the partition $\mathcal{P}$ to a new partition $\mathcal{Q}$, that is, moving the line of discontinuities S to a new line $S_\mathcal{Q}$ (contained in the set where F_ϵ is multidefined).

The extended map $F_\epsilon = \{f_{i,\epsilon} : U_i \mapsto B\}_i$, is now multidefined in $\bigcup_{i\neq j} U_i \cap U_j \supset S$. The separation property is an open condition, thus the extension F_ϵ still verifies $f_{i,\epsilon}(U_i) \cap f_{j,\epsilon}(U_j) = \emptyset$ for all $i \neq j$, if the neighborhoods U_i and U_j are chosen at a sufficiently small Hausdorff distance from their respective pieces B_i and B_j, and $\epsilon > 0$ is small enough.

Call $\epsilon_1 > 0$ to a positive real number smaller or equal than ϵ, and also smaller or equal than the distance from B_i to the complement of U_i, for all $i = 1, 2, \ldots m$. Precisely

$$0 < \epsilon_1 = \min\{\epsilon, \min_{1\leq i\leq m} \operatorname{dist}(B_i, U_i^c)\} \tag{31}$$

Consider the compact sets:

$$K^+ = \bigcap_{k\geq 1} \bigcup_{(i_1,\ldots,i_k)\in\{1,2\ldots m\}^k} f_{i_k,\epsilon} \circ \ldots \circ f_{i_1,\epsilon}(U_{i_1}) \quad \supset \quad K$$

$$K = \bigcap_{k\geq 1} \bigcup_{(i_1,\ldots,i_k)\in\{1,2\ldots m\}^k} f_{i_k} \circ \ldots \circ f_{i_1}(B_{i_1}) \tag{32}$$

Define the family $\mathcal{A}_{k,\epsilon}$ *of the extended atoms* of generation $k \geq 1$ for F_ϵ that form K^+, defined as follows:

The set $A \subset B$ is an extended atom of generation $k \geq 1$ if and only if there exists a word $(i_k, i_{k-1}, \ldots, i_1) \in \{1, 2, \ldots, m\}^k$ such that

$$A = f_{i_k,\epsilon} \circ \ldots \circ f_{i_1,\epsilon}(U_{i_1}).$$

The diameter of each extended atom of generation k is smaller that $diam(B) \cdot (\lambda')^k$ because $f_{i,\epsilon}$ is contractive with contraction rate $0 < \lambda' < 1$. Therefore, for sufficiently large $k \geq 1$ all the extended atoms of generation k that form K^+ have diameters smaller that $\epsilon_1/2$:

$$A \in \mathcal{A}_{k,\epsilon} \;\Rightarrow\; \operatorname{diam}(A) < \frac{\epsilon_1}{2}. \tag{33}$$

We assert that the extended atoms of generation $k \geq 1$ are pairwise disjoint: in fact, for two different $i \neq j$ the images are disjoint: $f_{i,\epsilon}(U_i) \cap$

$f_{j,\epsilon}(U_j) = \emptyset$. So the atoms of generation 1 are pairwise disjoint. Two extended atoms of generation $k \geq 1$ are $f_{i_k,\epsilon} \circ \ldots \circ f_{i_1,\epsilon}(U_{i_1})$ and $f_{j_k,\epsilon} \circ \ldots \circ f_{j_1,\epsilon}(U_{j_1})$. They can intersect if and only if $(i_1, i_2, \ldots, i_k) = (j_1, j_2, \ldots, j_k)$ because each $f_{i,\epsilon}$ is an homeomorphism onto its image. So, they intersect if and only if they coincide.

By construction, $U_i \supset B_i$ and $f_{i,\epsilon}|_{B_i} = f_i$. Therefore each of the atoms of generation k for F, is contained in the respective extended atom of generation k for F_ϵ, that has the same finite word $(i_1, i_2, \ldots, i_k)$.

If none of the extended atoms of generation k intersects S, then none of the atoms of generation k for F intersects S, and the system verifies the hypothesis of Lemma 4.2. So, in this case, there is nothing to prove, because F is finally periodic. (Recall our assumption at the beginning of this proof that the given F is not finally periodic.)

On the other hand, if some of the extended atoms of generation k intersects S, consider a new finite partition $\mathcal{Q} = \{C_i\}_{1 \leq i \leq m}$ of B such that the distance, defined in (26), between $\mathcal{Q}$ and the given partition $\mathcal{P}$ of F, is smaller than $\epsilon_1 > 0$:

$$\operatorname{dist}(\mathcal{P},\ \mathcal{Q}) < \epsilon_1 \leq \epsilon, \tag{34}$$

where $\epsilon_1 > 0$ was defined in the equality (31).

Choose the new partition $\mathcal{Q}$ such that the new separation line $S_{\mathcal{Q}} = \bigcup_{i \neq j}(C_i \cap C_j)$ does not intersect the extended atoms of generation k of K^+:

$$S_{\mathcal{Q}} \bigcap \left(\bigcup \{A \in \mathcal{A}_{k,\epsilon}\} \right) \ = \ \emptyset \tag{35}$$

This last condition is possible because the diameters of the extended atoms $A \in \mathcal{A}_{k,\epsilon}$ are all smaller than $\epsilon_1/2$, due to inequality (33). They are compact pairwise disjoint sets, because of the separation property. The distance between the two partitions $\mathcal{P}$ and $\mathcal{Q}$ is smaller than $\epsilon_1 > 0$ due to inequality (34), but can be chosen larger than $\epsilon_1/2$, and such that does not cut the atoms $A \in \mathcal{A}_{k,\epsilon}$, which verify inequality (33) and are all pairwise disjoint compact sets.

Due to the construction above and to the definition in equality (26), the maximum Hausdorff distance between the respective pieces B_i of $\mathcal{P}$ and C_i of $\mathcal{Q}$ is larger than $\epsilon_1/2 > 0$ and smaller than $\epsilon_1 > 0$.

We note that the old, and principally the new, separation lines $S_{\mathcal{P}}$ and $S_{\mathcal{Q}}$, are not necessarily C^1 nor even Lipschitz manifolds in the space B, and even if they are, they do not need to be ϵ_1- C^1 or Lipschitz near one from the other, to be ϵ_1 near with the Hausdorff distance.

The condition $\operatorname{dist}(\mathcal{P}, \mathcal{Q}) < \epsilon_1$ in (34), joined with the assumption $\operatorname{dist}(U_i^c, B_i) \geq \epsilon_1$ in (31, where B_i is the $i-$th piece of the partition $\mathcal{P}$, implies that the respective piece C_i of the partition $\mathcal{Q}$ verifies

$$C_i \subset U_i.$$

Therefore the extension $f_{i,\epsilon} : U_i \mapsto B$ in F_ϵ whose domain of definition is U_i can be restricted to C_i.

Define

$$G = \{g_i : C_i \mapsto B\}_{1 \leq m} \text{ where } g_i = f_{i,\epsilon}|_{C_i}.$$

By construction G and F coincide in $C_i \cap B_i$, the distance between the respective partitions $\mathcal{P}$ and $\mathcal{Q}$ is smaller than $\epsilon_1 \leq \epsilon$ due to (34), and the difference of their respective contraction rates λ' and λ is also smaller than ϵ, due to (30). So G is a ϵ-perturbation of the given F, according to the Definition 3.11.

It is enough now, to prove that G is finally periodic with persistent limit cycles.

Consider the limit set K_G of G as follows:

$$K_G = \bigcap_{k \geq 1} \bigcup_{(i_1,\ldots,i_k) \in \{1,2\ldots m\}^k} g_{i_k} \circ \ldots \circ g_{i_1}(C_{i_1})$$

As G is a restriction of F_ϵ to the sets $C_i \subset U_i$, we have that $K_G \subset K^+$, and in particular for all $k \geq 1$ the atoms of generation k for G, i.e. $g_{i_k} \circ \ldots \circ g_{i_1}(C_{i_1})$, are contained in the extended atoms of generation k for F_ϵ.

Due to inequality (35), the separation line $S_G = S_\mathcal{Q}$ among the continuity pieces C_i of G is disjoint with the extended atoms of generation k of F_ϵ. Therefore, it is also disjoint with the atoms of generation k of G. Then $G^k(B) \bigcap S_G = \emptyset$ and, applying lemma 4.2, G is finally periodic with persistent limit cycles. $\square$

It is possible (but not immediate) to construct, in a compact ball B of any dimension $n - 1 \geq 2$, piecewise continuous systems, uniformly locally contractive and with the separation property, as defined in Section 3, that do not verify the thesis of the Theorem 4.1, and thus their limit set is not composed only by periodic limit cycles, but it is a Cantor set attractor K defined by the Equality (32).

5 Discussion of the application to inhibitory neural networks.

The discontinuities of the Poincaré transformation F, due to spike phenomena in the neural network, play an essential role to study these systems, although it is an obstruction to apply mostly previously known results of the Topological Theory of Dynamics Systems, which is mostly developed for continuous dynamics.

We proved that in the generic stable case, the future limit orbits are all periodic: from all the initial states the system is given to limit cycles.

Due to the non-genericity of the bifurcating case, which is a consequence of Theorem 4.1, those non periodic dynamics would not robustly be seen in experiments: in fact, arbitrarily small perturbations in the parameters of the system will lead it to a periodic dynamics. These perturbations stabilize the system, to exhibit a limit set composed only by periodic cycles. As a consequence the bifurcating case appears only in the transition from one periodic behavior to other.

We also proved that the inter-spike interval is bounded away from zero for a positive time $T > 0$. It means that, generically, when the system is periodic, in spite of having preferred periodic patrons of discharges, the neurons do not synchronize in phase.

The last result implies that, if the number n of neurons in the system is very large, and if all of them are alive (they all spike along the period p), then the limit cycles of the network has a very large period $p \geq T \cdot n$, that may be even much larger than the observation time, or than the life time of the biological system. Therefore, in spite of being theoretically asymptotically periodic, these systems may not show its regularity. These two facts: extremely large periods, and irregularity inside the period, allow us to assert that those persistent systems with very large period p shall be in fact non-predictible for the experimenter, and will be perceived as virtually chaotic.

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