Synchronization of Many Dissipative Oscillators (Draft)

Propuesta de investigación en el
Seminario de Física No Lineal

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Synchronization of many dissipative oscillators.

Draft by Eleonora Catsigeras *

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To be discussed in a joint work with
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DRAFT ABSTRACT

We consider networks of any number of coupled oscillators. Each oscillator is mathematically modeled as an abstract dynamical system exhibiting a periodic attractor of the on-off type, that is dissipative during the off–phase and that interacts with the other oscillators during the on–phase. We **are trying to prove that those networks globally synchronize,** from Lebesgue almost all initial state, up to a positive error $\epsilon \ll 1$ in the measure of times and phases. We assume as hypothesis, that the couplings are all positive or excitatory, namely, they increase the velocities during the off-phases. We also assume that the oscillators in the network are identical, and completely mutually coupled, namely, all the oscillators are coupled with all the oscillators. The proofs are classical mathematical, and no numerical particular data is assumed.

NOTA: `Este draft sería un primer paso. El paso siguiente es tratar de probar la misma tesis de sincronización del teorema 9.1 para redes de osciladores no idénticos, pero` δ`-similares, y deducir que la sincronización es robusta, es decir, persiste ante perturbaciones pequeñas de los parámetros funcionales del modelo.`

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1 **Introduction**

`Redactar los primeros cuatro puntos y revisar o completar el quinto:`

$*$ Generalidades del problema de sincronización de osciladores que muestren su relevancia, con referencias bibliográficas

$*$ Problemas de sincronización ya resueltos o pendientes, en casos particulares de redes acopladas de osciladores del tipo On-Off, como por ejemplo los de los LCO, con referencias bibliográficas.

$*$ Open problem: to find mathematical sufficient conditions, and theoretical tools, to predict the synchronization of large networks of coupled oscillators.

$*$ Our contribution in this paper is ....

$*$ The methodology we use is the classical of pure mathematics, namely, the definitions, statement of theorems and their proofs are abstract, based on the classical logic and independent of particular numerical data and of experimental or computational experiments.

To state and prove the main result along this paper, in Theorem 9.1, predicting the global synchronization of large networks of coupled oscillators, we will consider a general abstract model of On-Off oscillator, defined in 3.1. We will

not restrict to any particular numerical example of On-Off oscillator,

but consider all of them, linear and non linear, with more or less regular or differentiable properties, in the abstract and wide scenario of all the infinitely many dynamical systems satisfying Definition 3.1. So, in this abstract scenario, **the parameters** are not a finite set of real numbers, but functional parameters, and thus, they

live in an infinite dimensional functional space.

Some of the ideas for the proof of Theorem 9.1, predicting the synchronization of the network of many identical oscillators, with complete and excitatory coupling, were taken from C. and Guiraud 2010. In that preprint we proved a similar theorem, but under different hypothesis: the differential equations determining the time evolution of each single oscillator were linear, the "On" states were assumed to be instantaneous. In this paper we are not assuming any concrete second term for the differential equations determining the evolution of each oscillator, and the "On" state is assumed to last a positive (ver small) time.

As said before, the results that we prove along this paper, hold independently of the numerical data of each particular example: it does not depend of the particular numerical formulae of the functions $F(x), G(x)$ and $H(x)$ in the second term of Functional of Differential Equations (1) and (4), that govern the dynamics of each oscillator and of the coupled network. About these functions, we only assume the open hypothesis of Inequalities (2) and (3). Nevertheless we assume a non open hypothesis: all the oscillators of the network are identical. It remains the open question of possibly extending some of the results in this paper, to networks of non identical oscillators, at least if they are δ-similar for some positive, small enough number δ.

2 A general abstract oscillator.

Definition 2.1 We call

phase dynamics in regime of an oscillator,

to any **continuous dynamical system** ϕ, depending of time $t \geq 0$ and on the initial condition $p_0 \in M$,

evolving in a metric space M, such that $M = \{\phi^t(p_0)\}_{t \geq 0}$

is a periodic orbit of period $T_0 > 0$.

In other words:

1) $\phi^t(p_0) = p_0 \quad \forall t \geq 0, \quad \forall\ p_0 \in M;$
2) $\phi^{t_1+t_2}(p_0) = \phi^{t_1}(\phi^{t_2}(p_0)) \quad \forall\ t_1 \geq 0, \quad \forall\ t_2 \geq 0, \quad \forall\ p_0 \in M;$
3) $\phi^{t+T_0}(p_0) = \phi^t(p_0) \quad \forall\ t \geq 0, \quad \forall\ p_0 \in M;$
4) T_0 is the minimum positive real number satisfying the equality 3); and
5) $\{\phi^t(p_0)\}_{0 \leq t < T_0} = M \quad \forall\ p_0 \in M.$

From 5) we deduce that for all $p_0, p_1 \in M$ there exists $t_1 = t_1(p_0, p_1)$ such that $p_1 = \phi^{t_1}(p_0)$.

We call *phase dynamics of the oscillator* to the dynamical system ϕ satisfying the conditions 1) to 5) above. Once an initial state p_0 is chosen and fixed, we call *instantaneous phase of the oscillator at time t,* or in brief *phase,* to the state $\phi^t(p_0)$.

REMARK: In particular, we are interested in some class of oscillators for which the phase dynamics

$\phi^t(p_0)$ is the flow that solves an autonomous ordinary differential equation, with initial condition $p_0 \in M$.

To do so, we need that the metric space M has a differentiable structure, for instance, M may be contained in the union $\widehat{M}$ of finite–dimensional differentiable Riemannian manifolds. If so, each integrable vector field $F(p)$ on the tangent manifold $T\widehat{M}$ defines a vectorial autonomous ordinary differentiable equation $dp/dt = F(p)$. The general solution of this differentiable equation is a dynamical system $\phi^t(p_0)$ evolving on $\widehat{M}$, called flow. This flow satisfies the conditions 1) and 2) of Definition 2.1. In particular if it has a periodic orbit of a point $p_0 \in \widehat{M}$, with period $T_0 > 0$, it also satisfies conditions 3) and 4). In general, we are interested to consider oscillators to all those dynamical systems defined as above, for which a periodic orbit $\{\phi^t(p_0)\}_{0 \leq t}$ is a topological or a Milnor attractor on the manifold $\widehat{M}$. If this were the case, we would restrict the space $\widehat{M}$ to $M \subset \widehat{M}$ defined as $M = \{\phi^t_{p_0}\}_{t \geq 0}$, so condition 5) of Definition 2.1 is also satisfied. In brief, the metric space M of Definition 2.1, even if being of one-dimensional (because it reduces to a single periodic orbit) may come from a periodic attractor (i.e. the dynamical system in the asymptotic regime) of a flow with many variables, which is, for instance, the solution of a vectorial ordinary differential equation on a manifold $\widehat{M} \supset M$ with many dimensions.

3 The On–Off dissipative oscillator.

Now, we will consider the On-Off oscillators, defined below, for which the metric space M is the union of two real intervals (or of two circles S^1):

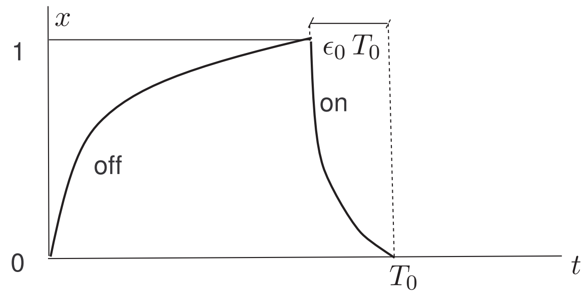

Figure 1: On-Off Relaxation Oscilator, with period T_0, and On-Time equal to $\epsilon_0\, T_0$

Definition 3.1 ON-OFF DISSIPATIVE OSCILLATORS (See Figure 1)

We say that an oscillator with phase dynamics ϕ *is On-Off* if the metric space M, where its dynamics in regime evolves, is:

$$M = \{(x, y) \in \mathbb{R}^2 : 0 \leq x < 1, y = 0\} \cup \{(x, y) \in \mathbb{R}^2 : 0 < x \leq 1, y = 1\}\ ,$$

and the phase dynamics ϕ in M is the continuous flow $\phi^t(x_0, y_0) = (x(t), y(t))$ that **solves the following system of functional equations** with initial conditions $x(0) = x_0, \;\; y(0) = y_0$:

$$\left\{ \begin{array}{lcl} \dfrac{dx}{dt} & = & (1-y)\ \cdot\ F(x)\ \ +\ \ y\ \cdot\ G(x) \\ y(t) & = & \left\{ \begin{array}{lllll} 0 & \text{if} & x(t) = 0 & \text{or if} & 0 < x(t) < 1 \text{ and } \lim_{\tau \to t^-} \dot{x}(\tau) > 0 \\ 1 & \text{if} & x(t) = 1 & \text{or if} & 0 < x(t) < 1 \text{ and } \lim_{\tau \to t^-} \dot{x}(\tau) < 0 \end{array} \right. \end{array} \right. \qquad (1)$$

where $F : [0, 1] \mapsto \mathbb{R}$ and $G : [0, 1] \mapsto \mathbb{R}$ are C^1 real functions such that

$$F(x) > 0, \quad F'(x) < 0, \quad G(x) < 0, \quad G'(x) < 0 \quad \forall\ x \in [0, 1]. \qquad (2)$$

For all the instantaneous phase states $(x(t), y(t))$ for which $y(t) = 0$ we say that the oscillator is ***Off.*** . For all the phase states for which $y(t) = 1$ we say that the oscillator is ***On.*** The variable y is called *the On-Off switch.* Notice, from the definition above, that $x(t)$ is strictly increasing satisfying $dx/dt = F(x(t)) > 0$ while the switch is Off, and it is strictly decreasing satisfying $dx/dt = G(x(t)) < 0$ while the switch is On.

We say that the oscillator above is ***dissipative,*** **because the kinetic energy $E_c(t) = (dx/dt)^2$ strictly decreases with time t,** during the Off face. In fact, we will argue for all t such that $y_i(t) = 0, 0 < x_i(t) < 1$:

$$\frac{dE_c(t)}{dt} = \frac{d\left(\frac{dx}{dt}\right)^2}{dt} = 2\frac{dx}{dt}\frac{d^2x}{dt^2} = 2F(x(t))\frac{d(\frac{dx}{dt})}{dt} = 2F(x(t))\frac{dF(x(t))}{dt} =$$
$$= 2F(x(t))F'(x(t))F(x(t)) = 2F^2(x(t)) \cdot F'(x(t)) < 0.$$

4 Existence and uniqueness of solution.

It is not immediate, but it is neither difficult, to prove that

the system of functional equations (1),

determining the time evolution in regime of a general abstract on-off oscillator,

has a unique solution for each initial condition $(x_0, y_0) \in M$.

Besides, all the solutions are the same orbit, which is periodic with period T_0. In fact, we prove this result in Proposition 4.1. Later, we give a concrete numerical example of an On-Off oscillator in 5.1.

Proposition 4.1 .
The system (1) *of functional equations in Definition* 3.1, *satisfies the following properties:*
(i) *For each initial condition* $(x_0, y_0) \in M$ *there exists a unique solution* $x(t), y(t)$ *of the system such that* $x(0) = x_0,\ y(0) = y_0$.
(ii) *All the solutions are defined for all* $t \geq 0$, *and are periodic with period* T_0 *which is independent of the initial condition.*
(iii) *There exists* $0 < \epsilon_0 < 1$, *independent of the initial condition, such that, during each periodic interval of time with length* T_0, *the oscillator is Off during a time subinterval of length* $(1-\epsilon_0)\cdot T_0$ *and is On during a time interval of length* $\epsilon_0 \cdot T_0$.

Proof: The first equation of the system (1) of functional equations in Definition 3.1, is an ordinary differential equation that does not satisfy the Piccard Theorem (because the second term has discontinuities). But for any given initial condition $(x_0, y_0) \in M$, and during the interval of time $0 \leq t \leq T_1$ while the value of y is constant $y(t) = y_0$, the first functional equation can be written as only one of the following two ordinary differential equations: $dx/dt = F(x)$, or $dx/dt = G(x)$. Both of them satisfy the hypothesis of Piccard Theorem, so there exists a unique solution $x(t), y(t)$ of the system, satisfying the given initial conditions and defined for all $t \in [0, T_1]$ where $T_1 > 0$. At time $t = T_1$, the switch variable y changes becoming $y(T_1) = y_1 \neq y_0$. This instant T_1 exists (is not infinite), because of the second functional equation: In fact, $y(t)$ must switch its value because $x(t)$ does arrive in a finite time T_1, to the level 1 or 0, since $x(t)$ is strictly monotone, with derivative that is bounded away from zero: $|dx/dt| \geq \min_{x\in[0,1]} \min\{|F(x)|, |G(x)|\}$.

Let us take now the initial condition $(x_1, y_1) = (x(T_1), y_1)$, for the instant $t = T_1$, instead of $t = 0$. Then we can apply the same argument as above to conclude that there exists a second interval of time $[T_1, T_2]$, with $T_2 > T_1$, and a unique solution $x(t), y(t)$ of the system (1), defined for all $t \in [T_1, T_2]$, and such that $y(t) = y_1$ for all $t \in [T_1, T_2]$ and $x(T_1) = x_1$. Arguing by induction, we deduce that there exists a unique solution $(x(t), y(t))$ satisfying (1) such that $x(0) = x_0,\ y(0) = y_0$. Besides, it is defined for all $t \in I = \bigcup_{i=0}^{+\infty} [T_i, T_{i+1}]$, where $T_0 = 0$.

To prove that the solution is defined for all $t \geq 0$ (namely $I = [0, +\infty)$), it is enough to prove that the orbit is periodic...

`pendiente: Terminar esta prueba.`

Definition 4.2 For an On-Off oscillator of period T_0, we call ***On–Time*** to the length of the temporal subinterval, inside each period, while the oscillator's phase is On. After Proposition 4.1 the On–Time **lasts $\epsilon_0 \cdot T_0$,** where $0 < \epsilon_0 < 1$.

In the sequel we will assume that ϵ_0 is much smaller than 1, and denote it as $\epsilon_0 \ll 1$.

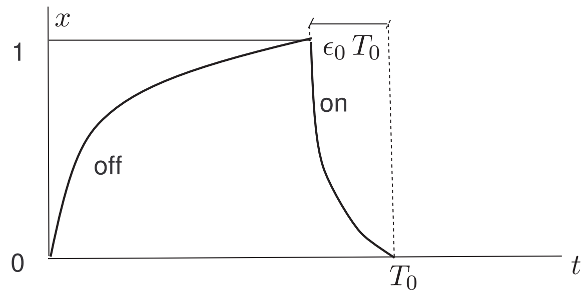

Figure 2: On-Off Relaxation Oscilator, with period T_0, and On-Time equal to $\epsilon_0\, T_0$

5 Example of on-off dissipative oscillator.

Example 5.1 Let us consider **an "On-Off Relaxation Oscillator",** for which the periodic evolution of the real variable x, as a function of the time t, is plotted in Figure 1.

This oscillator is defined on the metric space

$$M = \{(x, y) \in \mathbb{R}^2 : \quad 0 \le x < 1, \quad y = 0 \quad \text{or} \quad 0 < x \le 1, \quad y = 1\}.$$

The On-Off switch variable y, takes only the values 1 or 0, and is not plotted in Figure 1. It depends of the values of x and of the left lateral limit of dx/dt.

The limit cycle $\mathcal{O}$ of the oscillator in this example, is the periodic orbit of the dynamical system $(x(t), y(t))$, solution of the following functional equations:

$$\begin{cases} \dfrac{dx}{dt} & = & (1-y) \cdot \log 2 \cdot \dfrac{2-x}{T_0(1-\epsilon_0)} + y \cdot \log 2 \cdot \dfrac{-(1+x)}{\epsilon_0\, T_0} \\ \\ y(t) & = & \begin{cases} 0 & \text{if} \quad x(t) = 0 \quad \text{or if} \quad 0 < x(t) < 1 \text{ and } \lim_{\tau \to t^-} \dot{x}(\tau) > 0 \\ 1 & \text{if} \quad x(t) = 1 \quad \text{or if} \quad 0 < x(t) < 1 \text{ and } \lim_{\tau \to t^-} \dot{x}(\tau) < 0 \end{cases} \end{cases}$$

The periodic solution, taking the initial condition $x_0 = 0,\ y_0 = 0$, can be explicitly computed as follows, for all $t \ge 0$:

$$\begin{cases} x(t) & = & 2 - 2e^{\,(\log 2) \cdot t/(T_0(1-\epsilon_0))} \quad \text{if} \quad 0 \le t \le T_0(1-\epsilon_0) \\ x(t) & = & -1 + 2e^{-(\log 2) \cdot (t - T_0(1-\epsilon_0))/(\epsilon_0\, T_0))} \quad \text{if} \quad T_0(1-\epsilon_0) \le t \le T_0 \\ y(t) & = & 1 \quad \text{if} \quad 0 < t \le T_0(1-\epsilon_0) \\ y(t) & = & -1 \quad \text{if} \quad T_0(1-\epsilon_0) < t \le T_0 \\ x(t) & = & x(t+T_0) \quad \forall\, t \ge 0 \\ y(t) & = & y(t+T_0) \quad \forall\, t \ge 0. \end{cases}$$

6 Networks of oscillators.

Definition 6.1 *A network S of $n \geq 2$ on-off identical oscillators*
is a dynamical system

$$\phi^t(\mathbf{x}(0), \mathbf{y}(0)) = (\mathbf{x}(t), \mathbf{y}(t)),$$

evolving with time $t \geq 0$
on the product metric space M^n,
where M is the metric space defined in 3.1 for each single On-Off oscillator. In the notation above $(\mathbf{x}(t), \mathbf{y}(t)) \in M^n$ is the instantaneous state of the network at time t, computed as follows:

$$(\mathbf{x}, \mathbf{y}) = ((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)),$$

where, for all $i = 1, 2, \ldots, n$, the real variable $x_i(t) \in [0, 1]$ denotes the instantaneous phase of the i-th. oscillator of the network, and the variable $y_i(t) \in \{0, 1\}$ is its On-Off switch.

6.2 Uncoupled networks of identical on-off oscillators.

On one hand, **the dynamics of the network,** as Defined in 6.1, may be
the product dynamics of its
n-isolated **on-off oscillators,** each one given by the system satisfying Equations (1) and Inequalities (2). In this case we say that the oscillators of the network are all isolated, or that *the network is uncoupled.* For an uncoupled network of $n \geq 2$ on–off oscillators, the dynamics is governed by the a system of $2n$ -functional equations. In fact, for each value of $i \in \{1, 2, \ldots, n\})$ the variables $(x_i(t), y_i(t))$ describing the instantaneous state of the oscillator i are governes by the two functional equations given in Equalities (1).

On the other hand, in Definition 7.1 we will consider a class of networks for which some or all the oscillators, are mutually coupled. For that class, the dynamical system of the network is the solution of a system of vectorial functional equations, precisely composed with $2n$ real functional equations. Half of these functional equations are autonomous ordinary differential equations, and give conditions for the derivatives dx_i/dt of the phases $x_i(t)$ of all the oscillators $(i = 1, \ldots, n)$. The other half are functional equations governing the changes of the On–Off switch variables $y_i(t)$. In general, *the network of oscillators is said to be coupled* when some (not necessarily all) the differential equations governing the phases $x_i(t)$, write the derivative dx_i/dt of the phase of the $i - th$ oscillator, depending not only on $x_i(t)$ and $y_i(t)$, but also on the variables $(x_j(t), y_j(t))$ of some other oscillators of the network, namely, for some values of $j \neq i$. We are going to consider only some particular coupling as in the definition below.

7 Coupled Networks.

Definition 7.1 COUPLED NETWORKS OF ON-OFF OSCILLATORS.

Consider a network S of $n \geq 2$ identical on-off oscillators as defined in 6.1. We say that S ***is completely coupled with positive (or excitatory) interactions,***

if there exists a **C^1 function $H(x)$** defined for all $x \in [0,1]$, such that:

1)

$$H(x) > 0, \quad H'(x) < 0 \text{ for all } x \in [0,1]. \tag{3}$$

2) For all $i \in \{1, \dots, n\}$, the phase variable $x_i(t)$ of the $i-$th on–off oscillator of the network, and its on–off switch variable $y_i(t)$, satisfy the following system of functional equations:

$$\begin{cases} \dfrac{dx_i}{dt} = (1 - y_i) \cdot F(x_i) + y_i \cdot G(x_i) + \displaystyle\sum_{i \neq j, \; i,j=1}^{n} y_j \cdot H(x_i) \\ y_i(t) = \begin{cases} 0 & \text{if } x_i(t) = 0 \text{ or if } 0 < x_i(t) < 1 \text{ and } \lim_{\tau \to t^-} \dot{x}_i(\tau) > 0 \\ 1 & \text{if } x_i(t) = 1 \text{ or if } 0 < x_i(t) < 1 \text{ and } \lim_{\tau \to t^-} \dot{x}_i(\tau) < 0 \end{cases} \end{cases} \tag{4}$$

where $F : [0,1] \mapsto \mathbb{R}$ and $G : [0,1] \mapsto \mathbb{R}$ are the same C^1 real functions satisfying the Equations (1), for each i-th. oscillator. Therefore, we are still assuming that:

$$F(x) > 0, \quad F'(x) < 0, \quad G(x) < 0, \quad G'(x) < 0 \; \forall \; x \in [0,1]. \tag{5}$$

Definition 7.2 Minimum Positive Coupling ρ. Consider a coupled network of $n \geq 0$ on-off oscillators, as in Definition 7.1. *The Minimum Positive Coupling* is the positive real number, defined by:

$$\rho = \min_{x \in [0,1]} H(x) \; > \; 0. \tag{6}$$

Definition 7.3 Coupled network of always alive oscillators.

We say that an oscillator i of a coupled network with $n \geq 2$ on–off oscillators *eventually dies* for some initial condition $\mathbf{x}(0), \mathbf{y}(0) \in M^n$, if there exists $T_d \geq 0$ such that the switch variable $y_i(t)$ is constant for all $t \geq T$. We say that an oscillator i *is always alive* if for all initial condition it does not eventually die. We say that a coupled network with $n \geq 2$ on–off oscillators *is globally always alive,* if all its oscillators are always alive, according to Definition above.

Proposition 7.4 Global life of the network.

Consider a coupled network S of $n \geq 2$ identical on-off oscillators, with positive interactions, according with Definition 7.1. *If there exists a constant $M > 0$ such that*

$$G(x) + n(n-1) \cdot H(x) \leq -M < 0 \quad \forall \; x \in [0,1], \tag{7}$$

then, the network S is globally always alive.

Besides the interval of time in which any oscillator remains in the state "On" is upper bounded by

$$T^- = \frac{1}{M},$$

the maximum time in which any oscillator remains in the state "Off" is upper bounded by

$$T^+ = \frac{1}{\min_{x \in [0,1]} F(x)}.$$

Proof:

First assume that $y_i(\tau) = 1$, namely, the oscillator i is On at some instant $\tau \geq 0$. From Inequality (7) in the hypothesis, and applying the differential equation in (4), we deduce the following chain of assertions, for all $t \geq \tau$ such that $y_i(t) = 1$:

$$\frac{dx_i}{dt} = (1 - y_i(t)) \cdot F(x_i) + y_i(t) \cdot G(x_i) + \sum_{j \neq i} y_j(t) \cdot H(x_i)$$

$$\leq G(x_i) + n(n-1)H(x_i) \leq -M < 0.$$

In other words, the velocity dx_i/dt is negative and bounded away from zero. Therefore, the decreasing phase $x_i(t) \in [0,1]$, while $y_i(t) = 1$ arrives, after a finite time interval, to the lower threshold level 0. We deduce that the oscillator i turns to the Off phase, i.e. $y_i(t)$ takes the value 0 at some finite time $t > \tau$. Besides, the maximum time during which the oscillator i remains in the state “On”, occurs if $x_i(\tau) = 1$. We obtain that this maximum time is not greater than $T^- = 1/M$.

Similarly, assume that $y_i(\tau) = 0$, namely, the oscillator i is Off at some instant $\tau \geq 0$. After Weierstrass Theorem, there exists a minimum m of F in the compact interval $[0,1]$. Namely, $F(x_i) \geq m > 0$ for all $0 \leq x_i \leq 1$. Then, for all $t \geq \tau$ such that $y_i(t) = 0$, the differential equation in (4) implies:

$$\frac{dx_i}{dt} = F(x_i) + \sum_{j \neq i} y_j(t) \cdot H(x_i) \geq F(x_i) \geq m > 0.$$

In the last inequality we have used that the coupling function H is positive. In other words, the velocity dx_i/dt is positive and bounded away from zero. Therefore, the increasing phase $x_i(t) \in [0,1]$ arrives, after a finite time interval, to the upper threshold level 1, and the oscillator i turns to the On phase, i.e. $y_i(t)$ takes the value 1 for some $t > \tau$. Besides, the maximum time during which the oscillator i remains in the state “Off”, occurs if $x_i(\tau) = 0$. We obtain that this maximum time is not greater than $T^+ = 1/m$.

We have proved that, for all oscillator i, its On-Off switch variable $y_i(t)$ is not constant on $t \geq 0$. After Definition 7.3, the oscillator i is always alive. □

Remark 7.5 Consider a coupled network of $n \geq 0$ on-off oscillators, as in Definition 7.1, and satisfying Inequality (7). After Proposition 7.4, all the oscillators in the network are always alive. Besides, the maximum time in which it remains in the on state is upper bounded by $T^- = 1/M$. On the other hand, the same argument in the proof of Proposition 7.4, shows that the minimum time in which any oscillator remains in the off state, is lower bounded by

$$T'^{+} = \frac{1}{\max_{x \in [0,1]}(\ F(x) + n(n-1)H(x)\ )}.$$

Definition 7.6 **On/Off times relation γ.**

The *On/Off times relation* $\gamma > 0$ is the real number defined by:

$$\gamma := \frac{T^-}{T'^{+}} = \frac{\max_{x \in [0,1]}(\ F(x) + n(n-1)H(x)\ )}{\min_{x \in [0,1]}(\ |G(x)| - n(n-1)H(x)\ )}. \tag{8}$$

In the sequel we will assume that $\gamma \ll 1$.

8 Global Synchronization.

The following definition of the global synchronization phenomenon of all the oscillators of a network, is **indeed a quasi-synchronization.** We are imposing that **periodically all the oscillators exhibit their On states quasi-simultaneously** (up to **an error** on times upper bounded by a positive real number $\epsilon \ll 1$. But we are not hoping that they all arrive to the On-state, or leave the On-state simultaneously at exactly the same instants, nor that their instantaneous phases $x_i(t)$ coincide.

Definition 8.1 ϵ-global synchronization.
Let $0 < \epsilon \ll 1$ be a small positive real number.

We say that a network of $n \geq 2$ on-off oscillators ***ϵ-synchronizes at the instant t,*** if for all oscillator i there exists an instant

$$t_i \in [t, t + \epsilon]$$

such that i exhibits the "On"state at the time t_i, namely $y_i(t_i) = 1$.

We say that a network of $n \geq 2$ on-off oscillators ***eventually ϵ-synchronizes periodically,*** if there exist $t_0 \geq 0$ and $T > 0$ such that for all natural number $k \geq 0$,
the network ϵ-synchronizes at some instant in the time interval

$$[t_0 + kT, t_0 + (k + 1)T].$$

The time **t_0 is called *the transitory time*** to arrive to the ϵ-synchronization periodic synchronization, and the time
T is called *the period* of the network.

In Theorem 9.1 we predict the ϵ synchronization, as defined above, of networks of completely and positively coupled on-off identical oscillators, provided that they are dissipative. Nevertheless, in some cases of physical concrete examples with many oscillators, and depending on the initial states of the oscillators in the network, the transitory time t_0 may be very large, as much to make impossible to make observable in experiments the periodic synchronization. In fact, in Theorem 9.1 we find an optimal upper bound for the transitory time, that, depending on the parameters of the system, and of the initial state, that may take arbitrarily large values. So t_0 may be much larger than the reasonable time interval of observation.

9 The Synchronization Theorem.

Falta completar bien el enunciado del teorema que viene a continuación. Su enunciado va a depender de si andan o no, y cómo quedan al final, los intentos de pruebas de la sección 10, Cuando se trate de formalizar la ruta de prueba propuesta al final, puede derivar en cambios en las hipótesis, en las definiciones previas, y en las tesis.

Theorem 9.1 Up to now, still a conjecture

Let S be a network of $n \geq 2$ identical on–off dissipative oscillators, coupled positively and completely, accordingly with Definition 7.1. *Denote $\rho > 0$ to the minimum coupling constant defined in Equality* (6) *and $\gamma > 0$ the On-Off times relations defined in Equality* (8).

For all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\text{if} \quad \gamma < \delta \quad \text{and if} \quad ....\rho \;\; n \min_{x \in [0,1]} \min\{|F'(x)|, |G'(x)|\} \; etc \; etc \; > \; ...... ,$$

then, from Lebesgue–almost all initial state of the network, S eventually ϵ-synchronizes all its oscillators periodically (according to Definition 8.1). *Even more, the transitory time t_0 to arrive to the ϵ-synchronization, which depends on the initial condition of the network, and the period T of the synchronization of the network, are bounded by:*

$$t_0 \leq \frac{\rho....etc\ etc...f(\text{initial condition})}{n \;\; etcetc.... \cdot \max_{x \in [0,1]} |G(x)|} \quad .$$

$$....... \leq T \leq ....$$

See the attempt of proof of Theorem 9.1 in Section 10.

Remark: Hay que observar quizás, cuando se encuentre la fórmula precisa, que t_0 puede hacerse infinitamente largo para ciertas condiciones iniciales cercanas a aquellas con medida de Lebesgue cero para las cuales la red no sincroniza.

10 The proof of the Synchronization Theorem 9.1.

`Revisar si funcionan las ideas siguientes y en caso que funcionen, escribirlas bien; en caso que no funcionen buscar cómo modificar las hipótesis o las definiciones previas, o la tesis, para que funcionen, o buscar alguna ruta de prueba alternativa:`

$*$ **RUTA DE UNA POSIBLE PRUEBA DEL TEOREMA 9.1.**

To prove Theorem 9.1 we will first state some previous lemmas, which are rather technical:

Lemma 10.1 *Consider the constant times $T^+ > T'^+ \gg T^- > T'^- > 0$ defined as*

$$T^+ = 1 \ / \ (\min_{x \in [0,1]} F(x) \ ), \qquad etc, etc, etc. \tag{9}$$

then, for all $i \in \{1, 2, \ldots, n\}$ and for all instant $\tau \geq 0$ the following assertion holds:

If $y_i(\tau) = 0$ (i.e. the oscillator i is off at time τ), then there exists $\tau_2 \in (\tau, \tau + T^+]$, such that $(x_i(\tau_2), y_i(\tau_2)) = (1, 1)$. In other words, if the oscillator i is off, then it turns after a delay that is upper bounded by the constant T^+. If besides $x_i(\tau) = 0$, then $\tau_2 \geq \tau + T'^+$. In other words, the minimum time during which an oscillator is off is lower bounded by T'^+.

Idem if $y_i(\tau) = 1$ (i.e. the oscillator i is on at time τ), using the constants T^- and T'^-.

Proof: `Pendiente escribir bien el enunciado y la prueba, quizás solo como remark, en vez de lemma, pero todo junto. Fijarse que casi todo ya está probado en la proposición 7.4 y antes de definir el número` γ `en la igualdad (8).`

Lemma 10.2 *For all oscillators $i \neq j$, the following assertion holds:*
If $x_i(\tau) < x_j(\tau), y_i(\tau) = y_j(\tau)$ for some time $\tau \geq 0$, then
(1) *$x_i(t) < x_j(t)$ for all $t \geq 0$ such that $y_i(t) = y_j(t) = y_i(\tau)$, and*
(2) *$x_i(t) > x_j(t)$ for all $t \geq 0$ such that $y_i(t) = y_j(t) = 1 - y_i(\tau)$.*

Proof: `Prueba pendiente. Escribirla. Usar las ecuaciones diferenciales (4) y que los osciladores son idénticos. Fijarse en figura 1.`

Lemma 10.3 Contracting lemma.

Consider the constant T'^{+} *defined Lemma* 10.1. *Define the constant* $K > 0$ *as follows:*

$$K := \min_{x\in[0,1]} \min\{|F'(x)|, |G'(x)|\} = -\max_{x\in[0,1]} \max\{-F'(x), -G'(x)\}.$$

Then, for all oscillators $i \neq j$ *and for all interval of time* $[\tau, \tau + \Delta t]$ *with length* $\Delta t > 0$*, the following assertion holds:*

If $y_i(t) = y_j(t) \quad \forall \ \ t \in [\tau, \tau + \Delta t]$*, then*

$$|x_i(\tau + \Delta t) - x_j(\tau + \Delta t)| \leq |x_i(\tau) - x_j(\tau)| \cdot e^{-K\cdot\Delta t} < |x_i(\tau) - x_j(\tau)|.$$

Proof: Define $u(t) = x_i(t) - x_j(t)$. After Lemma 10.2, $u(t)$ has constant sign signo for all $t \in [\tau, \tau + \Delta t]$. It is not restrictive to assume that $u(t) > 0$. We now apply the differential equations in (4). In the case for which $y_i(t) = y_j(t) = 0$ we obtain

$$du/dt = A(t, x_i(t)) - A(t, x_j(t) \quad \text{where} \quad A(t, x) = F(x) + \sum_{k\neq i,j} y_k(t)H(x).$$

Since $A(t, x)$ is C^1 dependent on x, for all fixed t, we may apply the Lagrange Mean Value Theorem. We deduce that there exists a real function $\tilde{x}(t)$ defined for all $t \in [\tau, \tau + \Delta t]$, and that takes values in the interval $[x_j(t), x_i(t)]$ such that:

$$A(t, x_i(t)) - A(t, x_j(t)) = \left(F'(\tilde{x}(t)) + \sum_{k\neq i,j} y_k(t)H'(\tilde{x}(t) \right) \cdot (x_i(t) - x_j(t)) \leq$$

$$\leq (\max_{x\in[0,1]} F'(x)) \cdot u(t) \leq -K \cdot u(t) \quad \forall \ t \in [\tau, \tau + \Delta t].$$

In the last inequality we have used the hypothesis that $H'(x) < 0$. Therefore:

$$\frac{du}{dt} \leq -K \cdot u(t) \ \ \forall \ t \in [\tau, \tau + \Delta t].$$

Applying Gronwald Lemma:

$$u(t) \leq u(\tau) \cdot e^{-K\cdot t} \quad \forall \ t \in [\tau, \tau + \Delta t].$$

The proof in the case for which $y_i(t) = y_j(t) = 1$ is analogous. $\square$

Definition 10.4 **Switch-on times.** Denote $0 \leq t_1 \leq t_2 \leq t_3 \leq \ldots \leq t_k \leq t_{k+1} \leq \ldots$ to the sequence of *switch-on times,* namely:

$$\forall\ 1 \leq k \in \mathbb{N}\ \ \exists\ i \text{ such that } (x_i(t_k), y_i(t_k)) = (1,1),$$

$$\forall\ t \notin \{t_k\}_{k\geq 1}\ \ (x_i(t_k), y_i(t_k)) \neq (1,1)\ \ \forall\ i, \quad \text{and}$$

$$t_{k-1} < t_k = t_{k+1} = \ldots = t_{k+m-1} < t_{k+m} \quad \text{if} \quad \#\{i : (x_i(t_k), y_i(t_k)) = (1,1)\} = m.$$

From Proposition 7.4 we deduce that the sequence of switch-on times always exists and satisfies

$$\lim_{k\to+\infty} t_k = +\infty.$$

After Lemma 10.2, we can re-order the oscillators $\{1, 2, \ldots, n\}$, and re-index the sequence of switch times $\{t_k\}_{k\geq 1}$ such that:

$$0 \leq t_1 \leq t_2 \leq t_3 \leq \ldots \leq t_n \leq t_{n+1} \leq \ldots \text{ and}$$

$$\forall\ i \in \{1, 2, \ldots, n\} : \quad (x_i(t_i), y_i(t_i)) = (1,1) \quad \text{and} \quad (x_i(t_{i+kn}), y_i(t_{i+kn})) = (1,1)\ \ \forall\ k \in \mathbb{N}.$$

In the sequel we will assume that the set of oscillators is ordered as above. In other words: $i < j$ if and only if $t_i \leq t_j$ and this happens if and only if $t_{i+kn} \leq t_{j+kn}$ for all integer number $k \geq 0$.

Lemma 10.5 *For all given* $0 < \epsilon \ll 1$*, define the real number* $c > 0$ *by:*

$$c := \ldots.. \ll 1$$

Consider an oscillator i *and recall, from the construction of the switch on times that:*

$$(x_i(t_i), y_i(t_i)) = (1,1).$$

Then, for all oscillator $j > i$*:*

If $y_j(t_i) = 0$ *and* $1 - c \leq x_j(t_i) < 1$*, then* $(x_j(t_j), y_j(t_j)) = (1,1)$ *for* $t_i < t_j \leq t_i + \epsilon/4$.

Proof: `Pendiente, pero creo que es fácil.`

Lemma 10.6 *Recall the On/Off times relation* $\gamma > 0$ *defined in Equality* (8)*. Let* $0 < \epsilon \ll 1$ *be given. Consider the real numbers* $c > 0$ *defined in Lemma* 10.5*, and* $\delta > 0$ *defined by:*

$$\delta := \ldots.. \ll 1$$

If $\gamma < \delta$ *and if for some oscillators* $j > i$ *,* $t_i < t_j \leq t_i + \epsilon$ *then there exists* t'_j *such that* $t_j < t'_j \leq t_j + \epsilon$ *and*

$$(x_j(t'_j), y_j(t'_j)) = (0,0), \quad y_i(t'_j) = 0, \quad 0 < x_j(t'_j) < c.$$

Proof: `Pendiente.`

Lemma 10.7 *Recall the On/Off times relation $\gamma > 0$ defined in Equality (8). Recall the sequence $\{t_k\}_{k\geq 1}$ of switch-on times defined in 10.4. For all given $0 < \epsilon \ll 1$, define*

$$T = \ldots\ldots, \quad \delta = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Assume that $\gamma < \delta$ and that there exists $k > 1$ such that

$$t_k < \epsilon, \quad \text{and} \quad \forall\, i\ \exists\, t \in [0, t_k] \text{ such that } y_i(t) = 1.$$

Then the network $\epsilon-$ synchronizes *with period T and with transitory time $t_0 = 0$.*

Proof: `Pendiente. Idea: usar el lema de la contracción y los otros lemas ya enunciados.`

Remark: Consider the following assertion:

(A): For some instant t_i in the sequence of switch on time, for all oscillator j, either $y_j(t_i) = 1$ or $1 - c \leq x_j(t_i) < 1$.

After Lemmas 10.5 and 10.7, we deduce that if the assertion (A) were true, then the network ϵ synchronizes periodically with period T for all $t \geq t_i$.
So, **to end the proof of Theorem 9.1,** it is left to **prove (A) for Lebesgue almost all initial condition.**

(B): Assume by contradiction that for all switch-on time t_i there exists at least an oscillator j such that $y_j(t_i) = 0$ and $0 \leq x_j(t_i) < 1 - c$. Under this hypothesis, **denote**

$$M_i = \max_{j \neq i} \{\ x_j(t_i) \geq 0 : y_j(t_i) = 0, \quad x_j(t_i) < 1 - c\ \} < 1 - c.$$

Lemma 10.8 **The Key Lemma.**
There exists a constant $\lambda > 0$ such that, for Lebesgue almost all initial condition $\lambda < 1$, satisfying the following properties: Under the assumption (B) above, for all i there exists $j > i$ such that:

$$1 - M_j \ \leq\ \lambda \cdot (\ 1 - M_i\ )\ ,$$

Even more, one can choose

$$\lambda := \ldots\ldots f(\text{initial condition}).$$

Proof: `Pendiente. Espero que sea cierto porque este es el key point en la demostración. Hay que usar que la derivada segunda de la` F `y de las` H `son negativas, y/o el lema de la propiedad de contracción que demostré antes.`

10.9 End of the Proof of Theorem 9.1:

Assuming the contradiction hypothesis (B), and applying Lemma 10.8, there exists a natural number $k \geq 1$ such that $\lambda^k < c$. Therefore, for some fixed i, there exist $j_1 < j_2 < \ldots < j_k$, with $j_1 > i$, such that:

$$1 - M_{j_k} \ \leq \ \lambda^k \cdot (\ 1 - -M_i \ ) \ < c \cdot (\ 1 - M_i \ ) \leq c.$$

The last inequality implies that $M_{j_k} > 1 - c$ contradicting the definition of the sets M_j.

`Finalmente, hay que probar las cotas para el tiempo` t_0 `de transitorios, y para el período` T `se obtienen así:`
Applying Lemma 10.7:

$$T'^{+} + T'^{-} \leq T \leq T^{+} + T^{-} \ ,$$

where T^{+}, T^{-}, etc are the constants of Lemma 10.1.
The transitory time t_0 is by construction the switch on time t_{j_k} where $k \geq 0$ is the minimum natural number such that $\lambda^k < c$. It depends on the initial condition, but the worst case occurs if M_1 is near zero. So:

$$t_0 = t_{j_k} \leq k \cdot (T^{+} - T^{-}) \leq \frac{\log c}{\log \lambda}(T^{+} - T^{-}) = ... \quad \square$$

11 Conclusions

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References