Differential Geometry and Electromagnetism.

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Preface

These lecture notes are intended for a short course in mathematics focusing on the differential geometry of compact manifolds and the exterior Clifford algebra of differential forms, and their applications to the classical and relativist theories of electromagnetism in physics.

It is assumed that the reader is already familiar with the definitions of differential geometry, the exterior calculus, the algebra of differential forms and the integration theorems of differential forms on manifolds and submanifolds of euclidean spaces.

The notes are divided in four sections. Except the first one, which is an introduction to the whole work, each of the three other sections is independent from the others. Sections 1 and 2 are introductory. Sections 3 and 4 contain the main body of these notes.

In Section 1 we first include a brief historical note about the relation between the Clifford algebra of differential forms and the electromagnetism. Second, we introduce the Maxwell's Equations as a single condition for a closed 2- differential form in $\mathbb{R}^4$ have a given co-differential.

In Section 2 we fix the three first coordinates in the space $\mathbb{R}^4$ as spatial coordinates and the fourth coordinate as temporal coordinate. Then, we integrate the Maxwell's Equations on compact stationary manifolds of dimension 3, that are embedded in $\mathbb{R}^4$. In this way we prove the Gauss Theorem on the magnetic flow and the conservation law of electric charges, and other classic laws of electromagnetism.

In Section 3 we define the Lorenz transformations in $\mathbb{R}^4$ and deduce the proof of the relativity theorems of the Maxwell's Equations.

In Section 4 we fix an inertial referential system to traduce the classic electromagnetic problem of determining the vector fields that are generated by stationary charges and currents in the space, to the mathematical problem of computing an exact differential form knowing its co-differential (Poisson's Equation).

Finally, at the end of Section 4, we solve the Poisson's Equation by means of integration on a compact manifold of dimension 3, when the conditions along its frontier are known.

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1 Maxwell's Equations

1.1 Introduction

In 1968 Yvonne Choquet-Bruhat published her book [ChBr 1968] collecting the algebra of the exterior differential forms systems as a part with the Differential Geometry on manifolds. At one side, the Geometric Algebra had already been introduced 90 years before by William K. Clifford in [Cl 1878], as a generalization of the previous works of Grassman and Hamilton.

Since the decades of 1960 and 1970, the geometric Clifford algebra of differential forms, and the exterior calculus on manifolds (for precise definitios see for instance [Spi 1970] or also [Fle 1977, Chapter 7]) have been used exploiting their enormous power to unify many physical principles into one or two single theorems (see for instance [He-So 1992]).

A very general version of the Fundamental Theorem of Calculus had been proved, (see for instance [So-Sa 2011]), compiling in a unique statement many analogous theorems in various scientific disciplines. In particular its applications to Physics are studied rigorously in [Fla 1989], and more recently in [Fr 2012]. In brief, the formalization of the geometry applied to physical phenomena, provides the proofs of the basic principles, including, as a particular case, the proof of Maxwell's Equations of the electromagnetism. In fact, as we will see along this notes, these equations can be proved as corollaries of the abstract and general Stokes' Theorem.

Regarding the aspects of the applications to Engineering, until the decade of 1980, the differential forms of various degrees and the exterior calculus were not widely used (or even not much known) in applications to Electrical Engineering. Nevertheless, their relevance and simplifying power have been shown by providing natural mathematical associations with electromagnetic quantities. In fact, by using a single differential operator in the role of the familiar gradient, divergence and rotor operators, the differential algebra on manifolds provides an easy rigourous proof of the electromagnetic principles and, at the same time, a very concise method of solving theoretically electromagnetic problems (see for instance [De 1981]).

1.2 Notation

We fix the notation to be used along the four sections of these notes:

i) We denote by $C^\infty(\mathbb{R}^4, \mathbb{R})$ the space of all the real continuous functions $a : \mathbb{R}^4 \mapsto \mathbb{R}$ that have derivatives of all the orders and all the partial derivatives of any order are continuous.

ii) Any 2-differential form $\omega(a, b)$ in $\mathbb{R}^4$, when a coordinate system is given, is:

$$\omega(a, b) = (a_1\, dx^1 + a_2\, dx^2 + a_3\, dx^3) \wedge dx^4 + b_1\, dx^2 \wedge dx^3 +$$

$$+ b_2\, dx^3 \wedge dx^1 + b_3\, dx^1 \wedge dx^2,$$

where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are $3-$uples of functions of class $C^\infty(\mathbb{R}^4, \mathbb{R})$.

iii) We denote by $\omega_0^f, \omega_1^a, \omega_2^b, \omega_3^c$ the following 0, 1, 2 and 3-differential forms in $\mathbb{R}^4$:

- $\omega_0^f = f$, where $f \in C^\infty(\mathbb{R}^4, \mathbb{R})$
- $\omega_1^a = a_1\, dx^1 + a_2\, dx^2 + a_3\, dx^3$
- $\omega_2^b = b_1\, dx^2 \wedge dx^3 + b_2\, dx^3 \wedge dx^1 + b_3\, dx^1 \wedge dx^2$
- $\omega_3^c = c\, dx^1 \wedge dx^2 \wedge dx^3$, where $c \in C^\infty(\mathbb{R}^4, \mathbb{R})$.

iv) For each r fixed such that $0 \leq r \leq 3$, we denote by

$$C^\infty(\mathbb{R}^4, \Lambda^r(\mathbb{R}^4))$$

the set of r-differential forms in $\mathbb{R}^4$.

iv) We define *the differential operator d* from the set of r differential forms to the set of $r+1$ differential forms, as follows:

- $df := \omega_1^{\nabla f} + \partial f/\partial x^4\, dx^4$,
- $d\omega_1^a := \omega_2^{\nabla \wedge a} + \omega_1^{\partial a/\partial x^4} \wedge dx^4$,
- $d\omega_2^a := \omega_3^{\nabla \times a} + \omega_2^{\partial a/\partial x^4} \wedge dx^4$,

where we agree to denote:

the partial derivative $\partial a/\partial x^4 = (\partial a_1/\partial x^4, \partial a_2/\partial x^4, \partial a_3/\partial x^4)$,

the gradient $\nabla f = (\partial f/\partial x^1, \partial f/\partial x^2, \partial f/\partial x^3)$,

the rotor

$$\nabla \wedge\, a = A = \left(\begin{vmatrix} \partial/\partial x_2 & \partial/\partial x_3 \\ a_2 & a_3 \end{vmatrix}, \begin{vmatrix} \partial/\partial x_3 & \partial/\partial x_1 \\ a_3 & a_1 \end{vmatrix}, \begin{vmatrix} \partial/\partial x_1 & \partial/\partial x_2 \\ a_1 & a_2 \end{vmatrix} \right).$$

and finally, the divergence

$$\nabla \times a = \partial a_1/\partial x^1 + \partial a_2/\partial x^2 + \partial a_3/\partial x^3.$$

1.3 First Maxwell's Equations.

Theorem 1.3.1 *A 2-differential form $\omega(a,b)$ in $\mathbb{R}^4$ is closed* (i.e. $d\omega(a,b) = 0$) *if and only if the following two Maxwell's Equations hold:*

- $\nabla \wedge a + \partial b/\partial x^4 = 0$
- $\nabla \times b = 0$

Proof:

$\omega = \omega_1^a \wedge dx^4 + \omega_2^b$. Then:

$$d\omega = d\omega_1^a \wedge dx^4 + d\omega_2^b = \omega_2^{\nabla\wedge a} \wedge dx^4 + \omega_3^{\nabla\times b} + \omega_2^{\partial b/\partial x^4} \wedge dx^4 =$$

$$\omega_2^{\nabla\wedge a + \partial b/\partial x^4} \wedge dx^4 + \omega_3^{\nabla\times b}.$$

We have proved that $d\omega = 0$ if and only if $\nabla \wedge a + \partial b/\partial x^4 = 0$ and $\nabla \times b = 0$, as wanted. □

1.4 Inertial adjoint and co-differential forms

Consider the set $C^\infty(\mathbb{R}^4, \Lambda^r(\mathbb{R}^4))$ of $r-$ differential forms in $\mathbb{R}^4$, where $0 \le r \le 4$. Any $\omega \in C^\infty(\mathbb{R}^4, \Lambda^r(\mathbb{R}^4))$ can be written as:

$$\omega_{r-1}^a \wedge dx^4 + \omega_r^b,$$

where a y b are 3-uples of functions or functions, for each fixed r. We agree to define $\omega^a_{-1} = 0$.

Let $\# : C^\infty(\mathbb{R}^4, \Lambda^r(\mathbb{R}^4)) \mapsto C^\infty(\mathbb{R}^4, \Lambda^{4-r}(\mathbb{R}^4))$ be the isomorphism defined by

$$\#(\omega^a_{r-1} \wedge dx^4 + \omega^b_r) = -\omega^b_{3-r} \wedge dx^4 + (-1)^r \omega^a_{4-r}$$

$\#\omega$ is called the *inertial adjoint form* of ω. If ω in an r- differential form, then $\#\omega$ is a $(4-r)$-differential form.

We define the *inertial co-differential form* of an r- differential for ω in $\mathbb{R}^4$ $(1 \leq r \leq 4)$ to the $(r-1)$-form:

$$\overline{d}\omega = (-1)^{r+1} \# d \# \omega$$

Proposition 1.4.1 .
(Properties of the inertial co-differential form)

1) $\#\#\omega = (-1)^{r+1}\omega$

2) $\overline{d}\#\omega = \# d\omega$

3) $\#\overline{d}\omega = d\#\omega$

Proof: 1) Let us apply twice the definition of the operator # to the r-form $\omega = \omega^a_{r-1} \wedge dx^4 + \omega^b_r$:

$$\#\#\omega = \#(\#(\omega^a_{r-1} \wedge dx^4 + \omega^b_r)) =$$

$$= \#(-\omega^b_{3-r} \wedge dx^4 + (-1)^r \omega^a_{4-r}) =$$

$$= -(-1)^r \omega^a_{3-(4-r)} \wedge dx^4 - (-1)^r \omega^b_{4-(4-r)} =$$

$$= (-1)^{r+1}(\omega^a_{r-1} + \omega^b_r) = (-1)^{r+1}\omega,$$

proving Equality 1).

2) By definition of the operator $\overline{d}$ we have

$$\overline{d}\#\omega = (-1)^{r+1}\#d\#(\#\omega)$$

Thus, applying 1)

$$\overline{d}\#\omega = (-1)^{r+1}(-1)^{r+1}\#d\omega = \#d\omega,$$

proving Equality 2).

3) Let ω be an r-form. Then $\#\omega$ is a $(4-r)$-form and $d\#\omega$ is a $(4-r+1)$-form. We apply 1) to the $(3-r)$-form $d\#\omega$ to obtain

$$\#\#d\#\omega = (-1)^{4-r}d\#\omega = (-1)^{-r}d\#\omega$$

Then $d\#\omega = (-1)^r\#\#d\#\omega$, and applying the definition of the operator $\overline{d}$ to the r-form ω we conclude:

$$d\#\omega = (-1)^r\#\#d\#\omega = \#\overline{d}\omega,$$

ending the proof of Proposition 1.4.1. □

Remark: From Statement 2) of Proposition 1.4.1, note that the inertial co-differential operator $\overline{d}$ and the inertial adjoint operator # do not commute.

1.5 Second Maxwell's Equations

Let $\omega(a,b)$ be a 2-differential form in $\mathbb{R}^4$. Its inertial co-differential form is an 1-form which can generally be written as:

$$\eta(g,h) = gdx^4 - \omega_1^h,$$

where $g \in C^\infty(\mathbb{R}^4, \mathbb{R}), h = (h_1, h_2, h_3), h_i \in C^\infty(\mathbb{R}^4, \mathbb{R})$.

In the sequel, we denote $\eta(g,h)$ to the 1- form above in $\mathbb{R}^4$, while we still denote $\omega(a,b)$ to the given 2-form.

Theorem 1.5.1 *Let $\omega(a,b)$ be a 2-form in $\mathbb{R}^4$ and $\eta(g,h)$ be any 1-form also in $\mathbb{R}^4$. The sufficient and necessary conditions for $\overline{d}\omega(a,b) = \eta(g,h)$ are the following Second Maxwell's Equations:*

$$rot\ b - \partial a/\partial x^4 = h$$

$$\nabla \times a = g$$

Proof: First let us equivalent expressions of $\overline{d}\omega(a,b)$:

$$\overline{d}\omega(a,b) = -\#d\#(\omega_1^a \wedge dx^4 + \omega_2^b) = -\#d(-\omega_1^b \wedge dx^4 + \omega_2^a) =$$

$$-\#(-d\omega_1^b \wedge dx^4 + d\omega_2^a) = -\#(-\omega_2^{\nabla\wedge b} \wedge dx^4 + \omega_3^{\nabla\times a} + \omega_2^{\partial a/\partial x^4} \wedge dx^4) =$$

$$\#(\omega_2^{\nabla\wedge b - \partial a/\partial x^4} \wedge dx^4 - \omega_3^{\nabla\times a}) = \text{div } a \wedge dx^4 - \omega_1^{\nabla\wedge b - \partial a/\partial x^4}.$$

Then, we deduce that $\overline{d}\omega(a,b) = \eta(g,h)$ if and only if $\nabla \times a = g$ and $h = \text{ rot } b - \partial a/\partial x^4$, as wanted. $\square$

Definition 1.5.2 The electromagnetic vector fields

Up to some physical constants, the 3-uples of real functions a and b satisfying the Maxwell's Equations, are respectively called *electric vector field* and *magnetic vector field.* The real functions h and g are respectively called *electric current* and *electric charge.*

The 1- differential form $\eta(g,h)$ is called *generating form.*

When η is given, the problem that we will pose the Section 4 of these notes, consists in finding ω. We call this *the electromagnetic problem.* It is equivalent to the physical problem of determining the electromagnetic fields, generated by currents and charge in the space-time setting. Mathematically, it is translated to the problem of finding a closed 2-differential form in $\mathbb{R}^4$, when one knows its inertial co-differential.

2 The classic laws of electromagnetism.

2.1 Integral calculus on surfaces

Proposition 2.1.1 (Integral calculus on surfaces)

Let V be a differential surface in $\mathbb{R}^4$ (i.e. V is a 2-dimensional manifold) *in* $\mathbb{R}^4$.

Let $P_0 \in V$, and let (U,m) be a local map such that $P_0 \in U$.

Let P be the inverse function of m, i.e. $P : m(U) \mapsto U$, $P = P(x^1,x^2,x^3,x^4) \in U \subset V$, $x^i : m(U) \mapsto \mathbb{R}$. Let (u,v) be the coordinates in $m(U)$, and let $\{\partial/\partial u, \partial/\partial v\}$ be the (local) orientation in U.

Then
$$\int_u \omega_2^b = \iint_{m(U)} < b, P_u \times P_v > \, dudv,$$
where
$$P_u = \left(\frac{\partial x^1}{\partial u}, \frac{\partial x^2}{\partial u}, \frac{\partial x^3}{\partial u}\right), \quad P_v = \left(\frac{\partial x^1}{\partial v}, \frac{\partial x^2}{\partial v}, \frac{\partial x^3}{\partial v}\right),$$
and $P_u \times P_v$ denotes de vectorial product in $\mathbb{R}^3$ and $<, >$ the usual scalar product in $\mathbb{R}^3$.

Remark: This proposition implies that $\int_U \omega_2^b$ can be computed as the global normal flow of b through the surface U, with the orientation of the normal vector $N = P_u \times P_v$.
Proof: On the one hand, we have
$$\int_U \omega_2^b = \int_{m(U)} P^*(\omega_2^b) = \int_{m(U)} f \, du \wedge dv,$$
where we compute f applying $P^*(\omega_2^b)$ at the point (u, v) with the basis e_1, e_2 in $\mathbb{R}^2$. Then
$$f(u, v) = (P^*\omega_2^b)(u, v)(e_1, e_2) = \omega_2^b(P(u, v))(P_*e_1, P_*e_2),$$
where
$$P_* = \begin{pmatrix} \partial x^i/\partial u & \partial x^i/\partial v \\ \vdots & \vdots \end{pmatrix}$$
is a 4×2 dimensional matrix.

On the other hand:
$$\omega_2^b = b_1 \, dx^2 \wedge dx^3 + b_2 \, dx^3 \wedge dx^1 + b_3 \, dx^1 \wedge dx^2.$$

We obtain

$$\omega_2^b(Q)(v_1, v_2) = det \begin{pmatrix} b_1(Q) & b_2(Q) & b_3(Q) \\ v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \end{pmatrix}$$

The last equality can be checked after a straightforward computation. Then, we deduce:

$$f(u, v) = det \begin{pmatrix} b_1(P(u, v)) & b_3(P(u, v)) & b_3(P(u, v)) \\ \partial x^1/\partial u & \partial x^2/\partial u & \partial x^3/\partial u \\ \partial x^1/\partial v & \partial x^2/\partial v & \partial x^3/\partial v \end{pmatrix}$$

Finally, the last equality is translated as $< b(P(U, v)), P_u, P_v >$. $\square$

2.2 Gauss' Theorem

Lemma 2.2.1 *Let V be a r, $1 \leq r \leq 4$, dimensional, compact and orientable manifold in $\mathbb{R}^4$ with boundary ∂V. Let ω be a $5 - r$-differential form in $\mathbb{R}^4$. Then:*

$$\int_V \#\overline{d}\omega = \int_{\partial V} \#\omega. \tag{2.1}$$

Proof: From the equality $\#\overline{d}\omega = d\#\omega$, we obtain:

$$\int_V \#\overline{d}\omega = \int_V d\#\omega.$$

From Stokes' Theorem (see for instance [Spi 1970]) we know that

$$\int_V d\widehat{\omega} = \int_{\partial V} \widehat{\omega}'$$

for any $r-1$-differential form $\widehat{\omega}$. Defining $\widehat{\omega} = \#\omega$, and joining the last two equalities, we deduce (2.1), as wanted. □

Theorem 2.2.2 (Gauss' Theorem)

Let V be a 3-dimensional, compact and orientable manifold in $\mathbb{R}^4$, with boundary ∂V. Assume that V is contained in the 3-dimensional affine subspace $x^4 =$ constant. Then:

(1)

$$\int_{\partial V} \omega_2^b = 0,$$

(2)

$$\int_{\partial V} \omega_2^h = \frac{d}{dx^4} \int_V \omega_3^g.$$

Remark: (1) means that the magnetic flow through the frontier of V is null, at each fixed instant.

(2) means that the electric current flow through the frontier of V is equal to the variation of electric charge in its interior (electric charge conservation law).

Note: We orientate the boundary ∂V "to the exterior of V".

Proof of Theorem 2.2.2:

(1) $d\omega(a,b) = 0 \Rightarrow \int_V d\omega = \int_{\partial V} \omega(a,b) = 0$ due to Stokes' Theorem of differential calculus on manifolds. Then:

$$\omega(a,b) = \omega_1^a \wedge dx^4 + \omega_2^b,$$

and so

$$\int_{\partial V} \omega_1^a \wedge dx^4 + \int_{\partial V} \omega_2^b = 0.$$

(2) We have

$$\int_V \#\bar{d}\omega = \int_{\partial V} \#\omega$$

due to Lemma 2.2.1. Now, applying Theorem 1.5.1, we obtain:

$$\int_V \#\eta(g,h) = \int_{\partial V} \#\omega(a,b)$$

Substituing $\#\eta(g,h) = \omega_3^g + \omega_2^h \wedge dx^4, \quad \#\omega(a,b) = -\omega_1^b \wedge dx^4 + \omega_2^a$, and using that on the manifold V, by hypothesis, the equality $dx^4 = 0$ holds, we deduce:

$$\int_V \omega_3^g = -\int_{\partial V} \omega_2^a$$

in any fixed instant x^4. Taking derivatives with respect to x^4, and considering that V (and thus also ∂V) is contained in the affine subspace $X^4 = cte$, we obtain:

$$-\frac{d}{dx^4} \int_V \omega_3^g = \int_{\partial V} \omega_2^{\partial a/\partial x^4}.$$

Now we apply the second Maxwell's Equations to deduce:

$$\frac{\partial a}{\partial x^4} = \nabla \wedge b - h \quad \Rightarrow \quad \omega_2^{\partial a/\partial x^4} = d\omega_1^b - \omega_2^h,$$

where

$$\int_{\partial V} \omega_2^{\partial a/\partial x^4} = \int_{\partial V} d\omega_1^b - \int_{\partial V} \omega_2^h = \int_V dd\omega_1^b - \int_{\partial V} \omega_2^h$$

The first integral in the last member of the last equation is zero. Then:

$$\frac{d}{dx^4}\int_V \omega_3^g = \int_{\partial V} \omega_2^h.$$

$\square$

2.3 Other classic laws

i) We now integrate the equality

$$d\omega_1^a + \omega_2^{\partial b/\partial x^4} = 0$$

along a 2-dimensional manifold $S \subset \mathbb{R}^4$, which is stationary (i.e. S is contained in the affine subspace $x^4 =$ constant), compact and orientable. We obtain

Faraday Law:

$$\int_{\partial S} \omega_1^a = -\frac{d}{dx^4}\int_S \omega_2^b.$$

The equality obtained above is interpreted by the following statement:

The circulation of the electric field along the boundary of compact, orientable and stationary surface, is equal to the variation of the magnetic flow through the surface.

ii) Similarly, by integration of the equality

$$d\omega_1^b - \omega_2^{\partial a/\partial x^4} = \omega_2^h,$$

we obtain:

Biot-Savart Law:

$$\int_{\partial S} \omega_1^b = \int_S \omega_2^{h+\partial a/\partial x^4}.$$

iii) If besides a is independent of x^4, we deduce:

Ampére Law:

$$\int_{\partial S} \omega_1^b = \int_S \omega_2^h.$$

The last equality can be restated as follows:

If the electric field is stationary (i.e. independent of the time x_4), then, the circulation of the magnetic flow along the boundary of a compact, orientable surface, is equal to the electric current's flow through the surface.

3 Relativist Maxwell's Equations.

3.1 Lorenz transformations

Definition 3.1.1 Lorenz transformation

A linear transformation $\mathcal{L} : \mathbb{R}^4 \mapsto \mathbb{R}^4$, with associated matrix that we still denote as L, and with image point $Y = LX$, is *a Lorenz transformation* if

$$(y^1)^2 + (y^2)^2 + (y^3)^2 - (y^4)^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2$$

$\forall X = (x^1, x^2, x^3, x^4) \in \mathbb{R}^4, \ \ \forall Y = LX.$

Definition 3.1.2 Inertial System

With respect to a fixed coordinate system as a reference, other coordinate system is called *inertial,* if the change of coordinates' matrix has positive determinant and is a Lorenz transformation.

Remark: The Lorenz transformations do not necessarily preserve the usual metric of $\mathbb{R}^4$: they are not necessarily ortogonal transformations. Nevertheless, the subgroup of Lorenz transformations the leave invariant the fourth coordinate x^4 is formed by ortogonal transformations. This subgroup is isomorphic to the group of all isometric transformations in $\mathbb{R}^3$.

Theorem 3.1.3 .

(Associated matrix to a Lorenz transformation)

(1) *L is the associated matrix to a Lorenz transformation, if and only if* $L^T A L = A$, *where* $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

(2) If L is the matrix of change of variables from an inertial system to the reference system, then $\det L = 1$.

Proof: From the equality $Y = LX$, where L is a Lorenz transformation, we obtain

$$Y^T A Y = X^T A X \ \ \forall\, X \in \mathbb{R}^4.$$

Then

$$X^T (L^T A L) X = X^T A X \ \ \forall\, X.$$

In particular, for arbitrarily given X_1, X_2 in $\mathbb{R}^4$, we obtain:

$$(X_1 + X_2)^T (L^T AL)(X_1 + X_2) = (X_1 + X_2)^T A(X_1 + X_2)$$

After a short computation, and after canceling the terms that are the same on both sides of the equality, we obtain

$$X_2^T (L^T AL) X_1 + X_1^T (L^T AL) X_2 = X_2^T A X_1 + X_1^T A X_2.$$

Besides, from the following equality between real numbers

$$X_1^T (L^T AL) X_2 = (X_1^T (L^T AL) X_2)^T,$$

we deduce

$$X_1^T (L^T AL) X_2 = X_1^T A X_2 \;\; \forall \; X_1, X_2 \in \mathbb{R}^4.$$

By choosing the vectors X_1 and X_2 to belong to the basis of $\mathbb{R}^4$, we deduce that the entrances of the following two matrixes are the same: $L^T AL = A$.

The same computation is reversible. Therefore, if $L^T AL = A$ then L is a Lorenz transformation.

(2)

$$\det A = -1 = (\det L^T) \; detA \; detL = -(\det L)^2.$$

Then $\det L = \pm 1$, where $\det L > 0$. Therefore $\det L = 1$, as wanted. $\square$

3.2 First relativist Maxwell's equation

Consider the equality $f^*d\omega = df^*\omega$. We make any coordinates change $\mathcal{F}$ (the differential form ω expresses as $\mathcal{F}^*\omega$). The differentiation is uniquely defined, obtaining the same result if one changes de coordinate system before or after the differentiation. This implies that, if for some observer in the coordinates system (I) the differential form ω has differential η, then, for other observer in the coordinates system (II), the form $\mathcal{F}^*\omega$, corresponding to ω after the systems' change, has differential $\mathcal{F}^*\eta$, corresponding to η in the coordinates system (II).

Therefore, a closed differential form in a coordinates system will also be closed in any other coordinates system, being this last one an inertial system or not. An analogous assertion is also obtained for exact differential forms. Then, the properties of being closed or exact are intrinsic to the differential forms (i.e. they are properties that do not depend on the chosen coordinate system).

As an important difference with the differentiation, the inertial co-differential *does* depend on the coordinate system. Nevertheless, it is defined in such a way that it is invariant with all the inertial coordinates systems, with respect to the reference system. In relativist Physics, the only changes of coordinates that are under consideration, are those among inertial systems. Then, the co-differentiation is also an intrinsic operation to the differential forms, in the relativist sense.

Theorem 3.2.1 (First relativist Maxwell's equation)

Let α be a $2-$differential form in $\mathbb{R}^4$. In the reference coordinates system the necessary and sufficient condition to obtain $d\alpha = 0$ is that the so called first relativist Maxwell's equation holds:

$$\nabla \times b = 0$$

in all the inertial coordinate systems with respect to the reference one. In the equality above $\mathcal{L}^*\alpha = \omega(a, b)$, where $\mathcal{L}$ is the Lorenz transformation corresponding to the change of coordinates systems.

Proof:
The necessary condition: We have $d\alpha = 0 \ \Rightarrow d\mathcal{L}^*\alpha = 0$. From Theorem 1.3.1 we obtain $\nabla \times b = 0$. So, the necessary condition is proved.
The sufficient condition: For an inertial system, where the coordinates are (y^1, y^2, y^3, y^4), being $\mathcal{L}^*\alpha = \omega(a, b)$, we deduce:

$$d\omega = d\omega^1 \wedge dy^4 + \omega_2^{\partial b/\partial y^4} + \omega_3^{\nabla \times b}.$$

From the hypothesis $\nabla \times b = 0$. So, we obtain

$$\mathcal{L}^*(d\alpha) = d(\mathcal{L}^*\alpha) = \omega_2^M \wedge dy^4$$

In the reference system, the coordinates are x^1, x^2, x^3, x^4. In particular we have:

$$d\alpha = \omega_2^N \wedge dx^4, \quad N = (N_1, N_2, N_3), \quad N_i \in C^\infty(\mathbb{R}^4, \mathbb{R}).$$

Now, it is enough to prove that $N = 0$.

By contradiction, if $N \neq 0$, then some N_i is not null. To fix ideas assume that $N_1 \neq 0$. To end the proof, it is enough to find some inertial system such that in it $\nabla \times b$ is not null.

Let $\mathcal{L}$ be the Lorenz transformation, with any fixed $0 < v < 1$, which is given by the following equations:

$$x^1 = \frac{1}{\sqrt{1-v^2}}(y^1 - vy^4), \qquad x^2 = y^2$$

$$X^3 = y^3, \qquad x^4 = \frac{1}{\sqrt{1-v^2}}(y^4 - vy^1).$$

In the equalities above, after a short computation, one checks that $L^T AL = A$, where L is the associated matrix to the transformation $\mathcal{L}$, and besides $\det L = 1$).

From the computations above:

$$\mathcal{L}^*(d\alpha) = \mathcal{L}^*(\omega_2^N \wedge dx^4) = \omega_2^M \wedge dy^4.$$

Substituting the equations of the transformation $\mathcal{L}$ in the expression $\omega_2^N \wedge dx^4$, we obtain:

$$\omega_2^M \wedge dy^4 = \mathcal{L}^*(N_1 dx^2 \wedge dx^3 + (N_2 dx^3 - N_3 dx^2) \wedge dx^1) \wedge \mathcal{L}^* dx^4 =$$

$$= \omega_2^{M'} \wedge dy^4 - \frac{v}{\sqrt{1-v^2}} n_1 \circ \mathcal{L}(dy^2 \wedge dy^3 \wedge dy^1),$$

from where $M = M'$ and $N_1 = 0$ contradicting the assumption. $\square$

3.3 Pseudo inner product of differential forms in $\mathbb{R}^4$

To arrive to the second relativist Maxwell's equation, we previously need to prove that the inertial co-differential is a well defined operation, that is, it does not depend on the inertial coordinates system. To do that we need to define some mathematical tools.

Definition 3.3.1 Pseudo-Inner Product

Let ω y ω' be two r-differential forms in $\mathbb{R}^4$, such that

$$\omega = \omega^a_{r-1} \wedge dx^4 + \omega^b_3$$

$$\omega' = \omega^{a'}_{r-1} \wedge dx^4 + \omega^{b'}_3,$$

where a, a', b, b' three-uples or functions $C^\infty(\mathbb{R}^4, \mathbb{R})$, accordingly to the value of r. We define the *Pseudo Inner Product* $< \omega, \omega' >$ as follows:

$$< \omega, \omega' >= - < a, a' > + < b, b' >,$$

where $< a, a' >= aa'$ if they are real functions, or $< a, a' >= \sum_{i=1}^3 a_i a'_i$ if they are three-uples of real functions.

Remarks:

(1) $<,>$ is a transformation $<,>: C^\infty(R^4, \Lambda^r(\mathbb{R}^4)) \times C^\infty(R^4, \Lambda^r(\mathbb{R}^4)) \mapsto C^\infty(\mathbb{R}^4, \mathbb{R})$

(2) It is a bilinear and symmetric transformation. This assertion is easily checkable after its definition.

(3) The definition of $<,>$ is linked to the choice of the coordinates system.

Proposition 3.3.2 (Pseudo-inner product)
Let ω and ω' two $r-$ differential forms in $\mathbb{R}^4$ such that:

$$\omega = \alpha_1 \wedge \ldots \wedge \alpha_r, \quad \omega' = \beta_1 \wedge \ldots \wedge \beta_r,$$

where α_i, β_i are 1-differential forms in $\mathbb{R}^4$. Then:

$$< \omega, \omega' >= det(< \alpha_i, \beta_j >). \tag{3.1}$$

Proof: A basis of the space of all the $r-$ forms in $\mathbb{R}^4$ is

$$\{dx^{i_1} \wedge \ldots \wedge dx^{i_r}\}$$

where $1 \leq i_1 < \ldots < i_r \leq 4$.

We will first prove Equality (3.1) for ω and ω' belonging to the basis above, and then, we will prove the general case applying the linearity of the inner and outer products, and of the determinant.

(1) $\omega = dx^{i_1} \wedge \ldots \wedge dx^{i_r}, \quad \omega' = dx^{j_1} \wedge \ldots \wedge dx^{j_r}$ where $1 \leq i_1 < \ldots < i_r \leq 4$ and $1 \leq j_1 < \ldots < j_r \leq 4$.

We note that, in this case: $< \omega, \omega' >= 1$ if $i_1 = j_1, \ldots, i_h = j_h, \ldots, i_r = j_r < 4$, $\quad < \omega, \omega' >= -1$ if $i_1 = j_1, \ldots, i_h = j_h, \ldots, i_r = j_r = 4$, and $< \omega, \omega' >= 0$ in the other cases.

Besides, it is immediate that if $i_1 \neq j_1$, for instance if $i_1 < j_1$, then $i_1 < j_k \ \forall k : 1 \leq k \leq r$. Therefore we obtain

$$< dx^{i_1}, dx^{j_k} >= 0 \ \ \forall \ i_1 \neq j_k.$$

We deduce that the first row of the matrix $(< dx^{i_h}, dx^{j_k} >)_{i,j}$ is null, and thus, its determinant is zero.

If $i_1 = j_1, \dots, i_{h-1} = j_{h-1}, i_h \neq j_h$, for instance if $i_h < j_h$, then $i_h < j_k \ \ \forall h \leq k \leq r$. Besides, as $i_h > i_k = j_k$ if $1 \leq k < h$, we obtain $i_h \neq j_k$ if $1 \leq k \leq r, \quad < dx^{i_h}, dx^{j_k} >= 0 \quad \forall k$. Therefore, the h-th. row of the matrix is null and thus, its determinant is zero.

We conclude that either the determinant of the matrix is zero, or $i_h = j_h \ \ \forall 1 \leq h \leq r$.

But if $i_h = j_h$ then the matrix is diagonal, with all its terms equal to 1, except eventually the last term, which will be -1 if and only if $i_r = j_r = 4$.

Then, the determinant is equal to $< \omega, \omega' >$.

(2) Now, we will prove Equality (3.1) for

$$\omega = dx^{i_1} \wedge \ldots \wedge dx^{i_r}, \ \ 1 \leq i_1 \leq 4, 1 \leq i_h \leq 4, 1 \leq h \leq r.$$

On the one hand, if there exist repeated supra-indexes, then $\omega = 0$ and $< \omega, \omega' >= 0$. The respective rows in the matrix will be mutually equal and thus, the determinant is zero.

On the other hand, if there do not exist repeated supra-indexes, we permute them so ω becomes an $r-$ differential form multiplied by $(-1)^\sigma$ (where σ is the quantity of pair-transpositions of the needed permutation of the supra-indexes). In the corresponding matrix, for each transposition of a pair of supra-indexes, two rows are commuted. Then, its determinant is multiplied also by $(-1)^\sigma$. The proof of Equality (3.1) is now reduced to the proof of the case (1).

(3) Now, let ω' as in the case (2) and

$$\omega = dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_{r-1}} \wedge \alpha,$$

where $\alpha = \sum_{p=1}^{4} a_p \, dx^p$.

$$< \omega, \omega' >= \sum_{p=1}^{4} a_p \; < dx^{i_1} \wedge \ldots \wedge dx^p, \omega' >$$

After part (2), we obtain:

$$< \omega, \omega' >= \sum_{p=1}^{4} a_p \, det \begin{pmatrix} < dx^{i_1}, dx^{j_1} > & \ldots & < dx^{i_1}, dx^{j_r} > \\ \vdots & & \vdots \\ < dx^p, dx^{j_1} > & \ldots & < dx^p, dx^{j_r} > \end{pmatrix}$$

Due to the linearity of the determinant of the matrix, as a function of one of its rows, we can multiply by a_p the last row and sum all together. Applying the linearity of the inner product, that operation can be donde in a term inside the inner product. We deduce:

$$< \omega, \omega' >= \sum_{p=1}^{4} a_p \, det \begin{pmatrix} < dx^{i_1}, dx^{j_1} > & \ldots & < dx^{i_1}, dx^{j_r} > \\ \vdots & & \vdots \\ < \alpha, dx^{j_1} > & \ldots & < \alpha, dx^{j_r} > \end{pmatrix}$$

(4) The multi-linearity of $<,>$ and of the determinant allows to extend the method in the part (3) to prove Equality (3.1) in the general case. □

Theorem 3.3.3 .

(Intrinsic property of the pseudo-inner product)

The pseudo-inner product does not depend on the inertial coordinates. That is:

$$\mathcal{L}^* < \omega, \omega' >=< \mathcal{L}^*\omega, \mathcal{L}^*\omega' > \tag{3.2}$$

for all the Lorenz transformations $\mathcal{L}$ with positive determinant.

Proof: Let L be the matrix of the transformation $\mathcal{L}$, $X = LY$, where

$$L^T AL = A, \; detL = 1, \;\; A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

(1) In the reference system, let ω and ω' be the following 1-differential forms:

$$\omega = a_1 dx^1 + a_2 dx^2 + a_3 dx^3 + a_4 dx^4, \;\; \omega' = \sum_{i=1}^{4} a'_i dx^i.$$

In the inertial system we have:

$$\mathcal{L}^*(\omega) = \sum_{i=1}^{4} b_i \, {}^d y^i, \;\; \mathcal{L}^*(\omega') = \sum_{i=1}^{4} b'_i \, dy^i.$$

Let us denote

$$a(X) = \begin{pmatrix} a_1(X) \\ \vdots \\ a_4(X) \end{pmatrix}, \quad b(Y) = \begin{pmatrix} b_1(Y) \\ \vdots \\ b_4(Y) \end{pmatrix}.$$

We obtain

$$< \omega, \omega' >= [a(X)]^T A [a(X)].$$

Then

$$\mathcal{L}^* < \omega, \omega' >= [a(LY)]^T A [a(LY)].$$

On the other hand

$$< \mathcal{L}^*\omega, \mathcal{L}^*\omega' >= [b(Y)]^T A[b(Y)].$$

Besides, substituting in the formula

$$\omega = a_1 dx^1 + a_2 dx^2 + a_3 dx^3 + a_4 dx^4$$

the equation $X = LY$ of the basis change, we deduce

$$\mathcal{L}^*\omega = \sum_{i=1}^{4} b_i \, dy^i,$$

from where:

$$a(LY) = Lb(Y).$$

Therefore:

$$\mathcal{L}^* < \omega, \omega' >= [b(Y)]^T L^T AL[b(Y)].$$

Since $L^T AL = A$ the Equality (3.2) is proved for 1-differential forms.

(2) Let us prove Equality (3.2) in general, for all the $r-$ differential forms. Using Proposition 3.3.2, we can reduce to the case of 1-differential forms. In fact, any pair of $r-$differential forms ω and ω' can be written as follows:

$$\omega = \sum_{i=1}^{p} \alpha_{i,1} \wedge \alpha_{i,2} \wedge \ldots \wedge \alpha_{i,r} \quad \text{where } \alpha_{i,h} \text{ are } 1 - \text{forms},$$

$$\omega' = \sum_{j=1}^{q} \beta_{j,1} \wedge \beta_{j,2} \wedge \ldots \wedge \beta_{j,r} \quad \text{where } \beta_{j,k} \text{ are } 1 - \text{forms}.$$

Taking its pseudo-inner product, we obtain:

$$< \omega, \omega' >= \sum_{i=1}^{p} \sum_{j=1}^{q} < \alpha_{i,1} \wedge \ldots \alpha_{i,r}, \beta_{j,1} \wedge \ldots \beta_{j,r} >,$$

after the linearity of the inner product.

Applying Proposition 3.3.2, we obtain:

$$< \omega, \omega' >= \sum_{i=1}^{p} \sum_{j=1}^{q} det_{h,k}(< \alpha_{i,h}, \beta_{j,k} >).$$

Now, we apply the transformation $\mathcal{L}^*$, noting that it is linear. So:

$$\mathcal{L}^*(u \wedge v) = \mathcal{L}^* u \wedge \mathcal{L}^* v.$$

After the proof in case (1) and Proposition 3.3.2, we deduce

$$\mathcal{L}^* < \omega, \omega' >= \sum_{i=1}^{p} \sum_{j=1}^{q} det_{h,k}(< \mathcal{L}^* \alpha_{i,h}, \mathcal{L}^* \beta_{j,k} >) =$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q} < \mathcal{L}^* \alpha_{i,1} \wedge \ldots \wedge \mathcal{L}^* \alpha_{i,r}, \mathcal{L}^* \beta_{j,1} \wedge \ldots \wedge \mathcal{L}^* \beta_{j,r} >=$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q} < \mathcal{L}^*(\alpha_{i,1} \wedge \ldots \wedge \alpha_{i,r}), \mathcal{L}^*(\beta_{j,1} \wedge \ldots \wedge \beta_{j,r}) >=$$

$$=< \mathcal{L}^*(\sum_{i=1}^{p} \alpha_{i,1} \wedge \ldots \wedge \alpha_{i,r}), \mathcal{L}^*(\sum_{j=1}^{q} \beta_{j,1} \wedge \ldots \wedge \beta_{j,r}) >=< \mathcal{L}^* \omega, \mathcal{L}^* \omega' >,$$

ending the proof of Theorem 3.3.3. $\square$

3.4 On the Inertial Co-differential

Lemma 3.4.1 *Let ω be a $r-$ differential form in $\mathbb{R}^4$ and η be a $4-$ differential form in $\mathbb{R}^4$. Then:*

$$\omega \wedge \eta =< \#\omega, \eta > dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4.$$

Proof: On the one hand, we have

$$\omega = \omega^a_{r-1} \wedge dx^4 + \omega^b_r, \;\; \eta = \omega^{a'}_{3-r} \wedge dx^4 + \omega^{b'}_{4-r}, \;\; \#\omega = -\omega^b_{3-r} + (-1)^r \omega^a_{4-r}$$

and

$$< \#\omega, \eta >=< a', b > +(-1)^r < b', a > .$$

On the other hand we also have

$$\omega \wedge \eta = \omega^a_{r-1} \wedge dx^4 \wedge \omega^{b'}_{4-r} + \omega^b_r \wedge \omega^{a'}_{3-r} \wedge dx^4.$$

Substituting

$$\omega^a_{r-1}, \omega^{b'}_{4-r}, \omega^{a'}_{3-r}, \omega^b_r,$$

we obtain:

$$\omega \wedge \eta =< a, b' > (-1)^{4-r} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 +$$

$$+ < b', a > dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4,$$

ending the proof of Lemma 3.4.1. □

Theorem 3.4.2 .

(Intrinsic property of the inertial co-differential)

The inertial co-differentiation is independent of the inertial coordinates system. That is: $\overline{d}\mathcal{L}^*\omega = \mathcal{L}^*\overline{d}\omega$ for all the Lorenz transformations $\mathcal{L}$ with positive determinant.

Proof: Let ω be an $r-$ differential form and let η be a $(4-r)$-differential form. After Lemma 3.4.1 we obtain:

$$\mathcal{L}^*(\omega \wedge \eta) = \mathcal{L}^* < \#\omega, \eta >= \mathcal{L}^*(dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4)$$

where $\mathcal{L}$ is given by the equation $X = LY$.

$$\mathcal{L}^*(\omega \wedge \eta) =< \mathcal{L}^*\#\omega, \mathcal{L}^*\eta > detL(dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4)$$

where $\det L = 1$. As the pseudo inner product is well defined with respect to the inertial systems, we deduce:

$$\mathcal{L}^*(\omega \wedge \eta) =< \mathcal{L}^*\#\omega, \mathcal{L}^*\eta > dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4$$

Besides, after Lemma 3.4.1, we obtain:

$$\mathcal{L}^*(\omega \wedge \eta) = \mathcal{L}^*\omega \wedge \mathcal{L}^*\eta =$$

$$=< \#\mathcal{L}^*\omega, \mathcal{L}^*\eta > dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4.$$

Thus:

$$< \mathcal{L}^*\#\omega, \mathcal{L}^*\eta >=$$

$$=< \#\mathcal{L}^*\omega, \mathcal{L}^*\eta > \quad \forall \;\; r\text{-form } \omega, \quad \forall \;\; (4-r)\text{-form } \eta.$$

Choosing $\mathcal{L}^*\eta$ from the $(4-r)$-forms of the basis, we obtain that all the coordinates of $\mathcal{L}^*\#\omega$ and of $\#\mathcal{L}^*\omega$ are the same, from where $\mathcal{L}^*\#\omega = \#\mathcal{L}^*\omega$. Thus, the inertial adjoint is a well defined operation, defined the same for all inertial system. Therefore:

$$\overline{d}\omega = (-1)^{r+1}\#d\#\omega$$

$$\mathcal{L}^*\overline{d}\omega = (-1)^{r+1}\mathcal{L}^*\#d\#\omega = (-1)^{r+1}\#d\#\mathcal{L}^*\omega = \overline{d}\mathcal{L}^*\omega,$$

ending the proof of Theorem 3.4.2 $\square$

3.5 Second relativist Maxwell's equation.

Theorem 3.5.1 (Second relativist Maxwell's Equation) .

Let α be a 2-differential form and let β be a 1-differential form in $\mathbb{R}^4$.

Then, $\overline{d}\alpha = \beta$ in some reference coordinates system, if and only if the following second relativist Maxwell's equation *holds in all the inertial systems with respect to the reference:*

$$\nabla \times a = g,$$

where

$$\mathcal{L}^*\alpha = \omega(a, b),\ \mathcal{L}^*\beta = \eta(g, h),$$

and $\mathcal{L}$ is the Lorenz transformation of the coordinates change from the reference system to the inertial new system of coordinates.

Proof:

First, let us first the necessary condition:

$\overline{d}\alpha = \beta \Rightarrow \mathcal{L}^*\overline{\alpha} = \mathcal{L}^*\beta \ \Rightarrow \ \overline{d}(\mathcal{L}^*\alpha) = \mathcal{L}^*\beta \ \Rightarrow \ \nabla \times a = g$ after Theorem 1.5.1.

Second, let us prove the sufficient condition:

An inertial system in which the coordinates are denoted

$$y^1, y^2, y^3, y^4,$$

where $\mathcal{L}^*\alpha = \omega(a, b),\ \mathcal{L}^*\beta = \eta(g, h)$, satisfies

$$\mathcal{L}^*(\overline{d}\alpha - \beta) = \overline{d}(\mathcal{L}^*\alpha) - \mathcal{L}^*\beta =$$

$$\overline{d}\omega(a,b) - \eta(g,h) = \nabla \times a \;\; dy^4 - \omega_1^{\text{rot } b - \partial a/\partial y^4} - g dy^4 + \omega_1^h$$

After the hypothesis $\nabla \times a = g$, we obtain

$$\overline{d}\alpha - \beta = \omega_1^N, \quad N = (N_1, N_2, N_3), \; N_i \in C^\infty(\mathbb{R}^4, \mathbb{R}).$$

It is enough to prove that $N = 0$.

By contradiction, if $N \neq 0$, then some of the N_i, for instance N_1, is not zero: $N_1 \neq 0$. Let $\mathcal{L}$ be the Lorenz transformation with any fixed value $0 < v < 1$:

$$x^1 = \frac{1}{\sqrt{1-v^2}}(y^1 - vy^4), \quad x^2 = y^2,$$

$$x^3 = y^3, \quad x^4 = \frac{1}{\sqrt{1-v^2}}(y^4 - vy^1).$$

After a computation we deduce $L^T A L = A$ where L is the matrix of the transformation, satisfying $\det L = 1$. Therefore:

$$\mathcal{L}^*(\overline{d}\alpha - \beta) = \mathcal{L}^*(\omega_1^N) = \omega_1^M$$

$$\omega_1^M = \mathcal{L}^*(N_1 dx^1 + N_2 dx^2 + N_3 dx^3) = \omega_1^{M'} - \frac{N_1 \circ \mathcal{L} v}{\sqrt{1-v^2}} dy^4.$$

Thus $N_1 = 0$ contradicting our assumption. □

4 Determination of the electromagnetic field.

In this section we will work with stationary systems. By definition, a *stationary system* is that one where a, b, g, h are independent of x^4. Note that these systems live in the affine subspace of $\mathbb{R}^4$ obtained by the equation x^4 = constant. So, the differential forms are isomorphic to the forms in $C^\infty(\mathbb{R}^3, \Lambda(\mathbb{R}^3))$. In the stationary systems, once the Maxwell's equations are stated, we will study the problem of determining the electric and magnetic fields, when the electric currents and charges in the space are known given data. The electric currents and charges are given through what is called the *generating form.* To the purpose of determining the electromagnetic fields, we define some previous mathematical tools:

4.1 Ortogonal adjoint and co-differential in $\mathbb{R}^3$.

Using the same notations that were introduced at the beginning of Section 1, we have that $\omega_r^a,\ 0 \le r \le 3$ is an $r-$ differential form in $\mathbb{R}^3$. The system is stationary. So a is independent of x^4. Let

$$\omega_0^f = f \in C^\infty(\mathbb{R}^3, \mathbb{R}).$$

Definition 4.1.1 Orthogonal Adjoint

The *Orthogonal Adjoint* of ω_r^a in $\mathbb{R}^3$ is the $3-r$-differential form

$$*\omega_r^a = \omega_{3-r}^a.$$

Proposition 4.1.2 .

(Properties of the Orthogonal Adjoint)

(1) $*$ *is an isomorphism* $: C^\infty(\mathbb{R}^3, \Lambda^r(\mathbb{R}^3) \mapsto C^\infty(\mathbb{R}^3, \Lambda^{3-r}(\mathbb{R}^3)$.

(2) $* * \omega_r^a = \omega_r^a$.

(3) $*\psi\omega = \psi * \omega$, $\forall$ *0- form* ψ.

(4) $\omega_1^a \wedge (*\omega_2^b) = \omega_1^b \wedge (*\omega_2^a)$.

Proof: On the one hand (1), (2), y (3) are immediate consequences of Definition 4.1.1.

On the other hand, (4) is obtained from $\omega_1^a \wedge (*\omega_1^b) = \omega_1^a \wedge \omega_2^b = \omega_1^b \wedge \omega_2^a = \omega_1^b \wedge (*\omega_1^a)$. □

Definition 4.1.3 The Orthogonal Co-differential

The *Orthogonal Co-differential* of ω_1^a in $\mathbb{R}^3$, where $1 \leq r \leq 3$, is the following $(4-r)$-form:

$$\widetilde{d}\omega = *d * \omega$$

Proposition 4.1.4 .

(Properties of the Orthogonal Co-differential)

(1) $\widetilde{d} * \omega = *d\omega$.

(2) $*\widetilde{d}\omega = d * \omega$.

Proof: From Definitions 4.1.1 and 4.1.3 and Proposition 4.1.2 we obtain:

$$\widetilde{d} * \omega = *d * *\omega = *d\omega,$$

$$*\widetilde{d}\omega = * * d * \omega = d * \omega,$$

proving (1) and (2). $\square$

Remark: $*$ is the restriction to the forms in $\mathbb{R}^3$ of the operator, so called Hodge star.

More generally, the orthogonal co-differential in $\mathbb{R}^n$, for any natural number $n \geq 2$, is defined by

$$\widetilde{d}\omega = (-1)^{r(n-r)} * d * \omega,$$

where $*$ is called *the Hodge star operator.* In Definitions 4.1.1 and 4.1.3 are restricting the operators $*$ and $\widetilde{d}$ to $\mathbb{R}^3$.

The orthogonal co-differential $\widetilde{d}$ in $\mathbb{R}^4$ is, in its essence, different from the inertial co-differential $\overline{d}$ that was earlier defined in Section 3. In fact, the orthogonal co-differential $\widetilde{d}$ is invariant after orthogonal changes of coordinates in $\mathbb{R}^4$. But the inertial co-differential $\overline{d}$ is not necessarily invariant after all the orthogonal changes of coordinates. Nevertheless, as we proved in Section 3, $\overline{d}$ is invariant after inertial systems changes (Lorenz transformations with determinant equal to one).

Proposition 4.1.5 *In stationary systems, we define the operator # by:*

$$\#\omega := - * \omega_r^b \wedge dx^4 + (-1)^r * \omega_{r-1}^a, \tag{4.1}$$

where ω is the $r-$form in $\mathbb{R}^4$ defined by

$$\omega = \omega_{r-1}^a \wedge dx^4 + \omega_r^b.$$

Then:

$$\overline{d}\omega = \widetilde{d}\omega_{r-1}^a \wedge dx^4 - \widetilde{d}\omega_r^b. \tag{4.2}$$

Proof: Applying first the operator # and second the differential operator d, we obtain the following $(5-r)$-form in $\mathbb{R}^4$ for $1 \leq r \leq 4$:

$$d\#\omega = d\#(\omega^a_{r-1} \wedge dx^4 + \omega^b_r) = -d * \omega^b_r \wedge dx^4 + (-1)^r d * \omega^a_{r-1}$$

Now, applying Proposition 4.1.4 we deduce:

$$d\#\omega = -d * \omega^b_r \wedge dx^4 + (-1)^r d * \omega^a_{r-1} =$$

$$= - * \widetilde{d}\omega^b_r \wedge dx^4 + (-1)^r * \widetilde{d}\omega^a_{r-1}.$$

Then, by Identity 4.1 we obtain:

$$\#d\#\omega = -(-1)^r \widetilde{d}\omega^a_{r-1} \wedge dx^4 + (-1)^{5-r}(-1)\widetilde{d}\omega^b_r.$$

So, we conclude that

$$\overline{d}\omega = (-1)^{5-r}\#d\#\omega \quad \Rightarrow \quad \overline{d}\omega = \widetilde{d}\omega^a_{r-1} \wedge dx^4 - \widetilde{d}\omega^b_r,$$

ending the proof of Equality (4.2) as wanted. □

4.2 Electromagnetic problem in the stationary case

Proposition 4.2.1 *Under the stationary hypothesis, Maxwell's Equations are equivalent to the following ones:*

$$\widetilde{d}\omega^b_2 = \omega^h_1, \quad d\omega^b_2 = 0, \qquad \widetilde{d}\omega^a_1 = g, \qquad d\omega^a_1 = 0.$$

Proof: On the one hand we have:

$$d\omega(a,b) = 0 \;\Leftrightarrow\; d(\omega_1^a \wedge dx^4 + \omega_2^b = 0.$$

In particular, when the system is stationary, we obtain we following assertion

$$d\omega_1^a \wedge dx^4 + d\omega_2^b = 0 \;\Leftrightarrow\; d\omega_1^a = 0,\; d\omega_2^b = 0.$$

On the other hand we have

$$\overline{d}\omega(a,b) = \eta(g,h) \;\Leftrightarrow$$

$$\text{(in the stationary case) } \widetilde{d}\omega_1^a \wedge dx^4 - \widetilde{d}\omega_2^b = g dx^4 - \omega_1^h.$$

Thus, in the stationary case we conclude

$$\overline{d}\omega(a,b) = \eta(g,h) \;\Leftrightarrow\; \widetilde{d}\omega_1^a = g,\; \widetilde{d}\omega_2^b = \omega_1^h,$$

as wanted. □

The physical problem of finding the electromagnetic fields that are generated by currents and charges in the space, can be translated, under the stationary hypothesis, into the mathematical abstract problem of finding closed differential forms ω_2^b and ω_1^a in $\mathbb{R}^3$ when one knows their orthogonal co-differential.

4.3 Electrical and magnetic potentials and Poisson's Equation

If Maxwell's equations hold in an open star-shaped subset of $\mathbb{R}^3$, then by Poincaré's Lemma (see the statement and proof of Poincarés

Lemma for instance in [T-Z-H-L 2013]), the electromagnetic field is determined by

$$\omega_1^a = d\phi, \ \ \omega_2^b = d\omega_1^v$$

where ϕ is a 0-form that is called *the electrical potential* and v is a 3-uple of functions $C^\infty(\mathbb{R}^3, \mathbb{R})$ that is called *the magnetic potential vector.*

From Maxwell's Equations in the stationary case, we deduce that

$$\widetilde{d}d\phi = g, \ \ \widetilde{d}d\omega_1^v = \omega_1^h.$$

Thus, to solve the electromagnetic problem, it is enough to find ϕ and v satisfying the above equations.

The equation $\widetilde{d}d\phi = g$ is known as *Poisson's Equation.* It is a second order differential equation. In fact:

$$\widetilde{d}d\phi = *d * d\phi = *d * \omega_1^{\nabla\phi} =$$

$$= *d\omega_2^{\text{grad}_\phi} * \omega_3^{\text{div}(\text{grad } \phi)} = \text{ div (grad } \phi) = \Delta\phi,$$

where $\Delta\phi$ is called *the Laplacian of ϕ.* Thus, Poisson's Equation becomes

$$\Delta\phi = g \quad \text{where } \Delta\phi = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2}.$$

Lemma 4.3.1 .

If $v = (v_1, v_2, v_3)$, $v_i \in C^\infty(\mathbb{R}^3, \mathbb{R})$ then:

$$d\widetilde{d}\omega_1^v - \widetilde{d}d\omega_1^v = \sum_{i=1}^{3} \Delta v_i dx^i$$

Proof: First, we apply twice the definition of rot$v$. After an immediate computation we obtain:

$$\text{rot}(\text{ rot } v) = \text{ grad } (\text{div } v) - (\Delta v_1, \Delta v_2, \Delta v_3).$$

Then:

$$d\widetilde{d}\omega_1^v - \widetilde{d}d\omega_1^v = d * d\omega_2^v - *d * \omega_2^{\nabla\wedge v} =$$

$$= d * \omega_3^{\nabla\times v} - *d\omega_1^{\text{rot } v)} = d(\nabla \times v) - \omega_2^{\text{rot } (\text{rot } v} =$$

$$= \omega_1^{\text{grad}(\text{div } v)} - \omega_1^{\text{rot } (\text{rot } v)} =$$

$$= \omega_1^{(\Delta v_1, \Delta v_2, \Delta v_3)} = \sum_{i=1}^{3} \Delta v_i dx^i.$$

$\square$

4.4 On the magnetic potential.

In an open star-shaped set, due to Poincaré lemma, we immediately obtain that:

$$d\omega_1^v = d\omega_1^{v'} \ \Leftrightarrow\ v = v' + \text{ grad } \psi.$$

Then, to determine all the possible magnetic potentials, it is enough to determine a particular potential vector v. Therefore, one tries to find some particular solution of the differential equation

$$\widetilde{d}d\omega_1^v = \omega_1^h.$$

Theorem 4.4.1 .

If $\widetilde{d}\omega_1^v = 0$ *then*

$$\widetilde{d}d\omega_1^v = \omega_1^h \;\Leftrightarrow\; \Delta v_i = h_i$$

Proof: From Lemma 4.3.1 we have

$$d\widetilde{d}\omega_1^v - \widetilde{d}d\omega_1^v = \sum_{i=1}^{3} \Delta v_i dx^i,$$

proving the theorem. □

Conclusion: To solve the electromagnetic problem it is enough to solve the following system of two Poisson's Equations:

$$\Delta\phi = g, \quad \Delta v_i = h_i.$$

4.5 Integration Theorems

We call the homogeneous Poisson's Equation *Laplace Equation*; namely:

$$\Delta\phi = 0.$$

For example, consider the real function $d(P) = d(P, P_0)$ in $\mathbb{R}^3$, where P_0 is a fixed point. Thus,

$$1/d \in C^\infty(\mathbb{R}^3 - \{P_0\}, \mathbb{R}).$$

It is standard to check that $1/d$ is a solution of the Laplace Equation.

Lemma 4.5.1 .

If V is a compact orientable manifold of dimension $r, 1 \leq r \leq 3$ in $\mathbb{R}^3$, with boundary ∂V, and if ω is a $4-r$-form in V, then

$$\int_V *\widetilde{d}\omega = \int_{\partial V} *\omega.$$

Proof: It is enough to apply Stokes's Theorem of the differential calculus on manifolds, taking into account that $*\widetilde{d}\omega = d * \omega$. □

Theorem 4.5.2 .

Let ψ, ϕ two functions in $C^\infty(\mathbb{R}^3, \mathbb{R})$. If V is a compact orientable manifold of dimension 3 in $\mathbb{R}^3$, with boundary ∂V, then:

$$\int_V *(-\psi\widetilde{d}d\phi + \phi\widetilde{d}d\psi = \int_{\partial V} -\psi(*d\phi) + \phi(*d\psi). \tag{4.3}$$

Proof: Construct $\omega := \psi d\phi - \phi d\psi$. Then ω is a $1-$form in $\mathbb{R}^3$. On the one hand, we have

$$*\omega = \psi * d\phi - \phi * d\psi.$$

On the other hand, the following equality holds:

$$*\widetilde{d}\omega = d * \omega = d(\psi * d\phi - \phi * d\psi) =$$

$$= d\psi \wedge *d\phi - d\phi \wedge *d\psi + \psi d * d\phi - \phi d * d\psi.$$

After the proof of part (4) of Proposition 4.1.2 (that states the properties of the orthogonal adjoint), the first two terms of the sum at right in the above equality are zero. Then:

$$*\widetilde{d}\omega = \psi * \widetilde{d}d\phi - \phi * \widetilde{d}d\psi.$$

Now, applying the previous Lemma 4.5.1, we deduce Equality (4.3), ending the proof. □

4.6 Solution of Poisson's Equation

Let us search for solutions ϕ of the Poisson's Equation

$$\Delta\phi = g,$$

on a compact orientable manifold V of dimension 3 with boundary, provided on knows the data of the boundary conditions for ϕ.

Let $P_0 \in V$, $P_0 \notin \partial V$. To fix ideas we assume that P_0 is the origin $(0,0,0)$. If not, we just make a translation of ϕ.

Since $P_0 \notin \partial V$, then P_0 is in the interior V. Thus, we can consider an open ball $E \subset \mathbb{R}^3$, of radius $\epsilon > 0$ and centre at P_0, which is contained in V.

Let V' the new manifold with boundary defined by

$$V' = V \cap E^c,$$

where E^c is the complement of E in $\mathbb{R}^3$. Its boundary $\partial V'$ is the union of the boundary ∂V of the given manifold V, with the 2-dimensional surface σ which is the boundary of the ball E. We orientate σ with the normal vector towards the centre P_0 of the ball. Applying the above Theorem to the manifold V', and taking $\psi = 1/r$, where $r(P) = d(P, P_0)$, we obtain $\psi \in C^\infty(V, \mathbb{R})$, which is a solution of Laplace's Equation. Then:

$$\int_{V'} *(1/r)\widetilde{d}d\phi + \int_{\partial V} \phi * d(1/r) - (1/r) * d\phi =$$

$$= \int_\Sigma (1/r) * d\phi + \int_\Sigma \phi * d(1/r). \qquad (4.4)$$

Now, let us study the two integrals in the second term of the above equality:

(a) Since Σ is the spherical surface with radius ϵ we have

$$\int_\Sigma (1/r) * d\phi = (1/\epsilon) \int_\Sigma *d\phi,$$

where ϕ is defined in the whole ball E. By the Stokes Theorem of the differential calculus on manifolds, we deduce

$$\int_\Sigma *d\phi = -\int_E d * d\phi =$$

$$= -\int_E *\widetilde{d}d\phi = -\int_E \Delta\phi dx \wedge dy \wedge dz.$$

Since $\Delta\phi$ is continuous, because ϕ is C^∞ in the compact ball $\overline{E}$ we obtain $|\Delta\phi| < M$ in E, for some real constant $M > 0$. Thus:

$$\Big| \int_E *d\phi \Big| < M \int_E dx \wedge dy \wedge dz = \frac{4}{3}\pi M \epsilon^2,$$

from where we conclude that

$$\Big| \int_\Sigma \frac{1}{r}(*d\phi) \Big| < \frac{4}{3}\pi M \epsilon^2. \qquad (4.5)$$

(b) Now, let us compute $\int_\Sigma \phi * d(1/r)$, where $r = \sqrt{x^{1^2} + x^{2^2} + x^{3^2}}$. In fact:

$$d\frac{1}{r} = \sum_{i=1}^{3} \frac{\partial \frac{1}{r}}{\partial x^i} dx^i =$$

$$= -\frac{1}{r^2}\sum_{i=1}^{3}\frac{\partial r}{\partial x^i}dx^i = -\frac{1}{r^2}\omega_1^{\nabla r}.$$

Thus:

$$*d\frac{1}{r} = -\frac{1}{r^2}\omega_2^{\nabla r}, \quad \int_\Sigma \phi * d\frac{1}{r} =$$

$$= -\frac{1}{\epsilon^2}\int_\Sigma \omega_2^{\phi\nabla r}.$$

After Proposition 2.1.1, this last integral coincides with the flow of ϕ grad r through the spherical surface Σ oriented in a such way that its normal vector points towards the centre P_0 of the ball E.

If we take parameters of the surface $P(u,v) = (x^1(u,v), x^2(u,v), x^3(u,v))$: $x^1 = \epsilon \operatorname{sen} u \cos v$, $x^2 = \epsilon \operatorname{sen} u \operatorname{sen} v$, $x^3 = \epsilon \cos u$, $P : (-\pi,\pi) \times (0, 2\pi) \mapsto \Sigma$, we deduce that:

$$\int_\Sigma \omega_2^{\phi \text{ grad } r} = \int_{(-\pi,\pi)\times(0,2\pi)} < \phi \text{ grad } r, P_u \times P_v > \, dudv.$$

Besides,

$$r = \sqrt{x^{1^2} + x^{2^2} + x^{3^2}}.$$

So

$$\partial r/\partial x^i = x^i/r, \quad \text{grad } r = (x^1/r, x^2/r, x^3/r).$$

On the spherical surface Σ the following equalities hold:

$$\text{grad } r = (\operatorname{sen} u \cos v, \operatorname{sen} u \operatorname{sen} v, \cos u).$$

Then we obtain the following equalities for the inner product of grad$r$ and $P_u \times P_v$:

$$< \text{ grad } r, P_u \times P_v >= det \begin{pmatrix} \operatorname{sen} u \cos v & \operatorname{sen} u \operatorname{sen} v & \cos u \\ \epsilon \cos u \cos v & \epsilon \cos u \operatorname{sen} v & -\epsilon \operatorname{sen} u \\ -\epsilon \operatorname{sen} u \operatorname{sen} v & \epsilon \operatorname{sen} u \cos v & 0 \end{pmatrix}$$

$$= \epsilon^2 \operatorname{sen} u.$$

Therefore:

$$\int_\Sigma \omega_2^{\phi \text{ grad } r} = -\epsilon^2 \int_{-1}^{1} dt \int_0^{2\pi} \phi \, dv.$$

We obtain:

$$\int_\Sigma \phi * d\frac{1}{r} = \int_{-1}^{1} dt \int_0^{2\pi} \phi \, dv.$$

By the Theorem of the mean value of integral calculus, there exist t_ϵ and v_ϵ such that

$$\phi(P(t_\epsilon, v_\epsilon)) = \frac{\displaystyle\int_{-1}^{1} dt \int_0^{2\pi} \phi \, dv}{\displaystyle\int_{-1}^{1} dt \int_0^{2\pi} dv} = \frac{1}{4\pi} \int_\Sigma \phi * d(1/r).$$

Then, for each fixed $\epsilon > 0$ there exists a point P_ϵ of the spherical surface Σ satisfying the following equality:

$$\int_\Sigma \phi * d(1/r) = 4\pi\phi(P_\epsilon).$$

Now we substitute the formulae of parts (a) and (b) in Equality 4.4, to deduce that:

$$0 \leq \Big| \int_{V'} *(1/r)\widetilde{d}d\phi + \int_{\partial V} (\phi * d(1/r) - (1/r) * d\phi) - \phi(P_\epsilon)4\pi \Big| <$$

$$< \frac{4}{3}\, \pi \, M \, \epsilon^2.$$

Since this inequality holds for all $\epsilon > 0$ such that $S_\epsilon \subset V$, we can take $\epsilon \to 0^+$ to obtain the following equality:

$$\int_V *(1/r)\widetilde{d}d\phi + \int_{\partial V} (\phi * d(1/r) - (1/r) * d\phi) = \phi(P_0)4\pi,$$

from where:

$$\phi(P_0) = \int_V * \frac{\widetilde{d}d\phi}{4\pi r} + \frac{1}{4\pi} \int_{\partial V} (\phi * d(1/r) - (1/r) * d\phi).$$

By Poisson Equation $\widetilde{d}d\phi = g$, where g is a given function. Then, the formula of the solution $\phi(P_0)$ in any point P_0 in the interior of V, when one knows the function ϕ and its diferencial on the boundary surface ∂V, becomes:

$$\phi(P_0) = \int_V \frac{g}{4\pi r} + \frac{1}{4\pi} \int_{\partial V} (\phi * d(1/r) - (1/r) * d\phi).$$

We have found the solution ϕ, as wanted.

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