Curvatura de curvas planas

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Cálculo 3

IMERL

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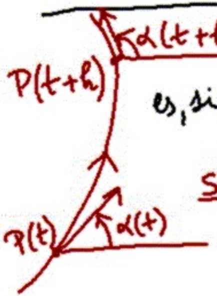
[1] Sobre notas de Eleonora Catsigeras

definición curvatura

CURVATURA de CURVAS PLANAS

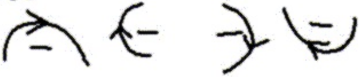


DEF. Curvatura $K(t)$ en el punto $P(t)$ es, si e $\exists$,

$$\boxed{\lim_{h\to 0} \frac{\alpha(t+h)-\alpha(t)}{s(t+h)-s(t)}} = \frac{\dot\alpha(t)}{\dot s(t)}$$

SIGNO $K(t)>0$ dobla hacia la izquierda

$K(t)<0$ dobla hacia la derecha



K es INTRÍNSECA

curvatura de una curva plana

$s = s(t)$

$$\vec{P}' = \frac{dP}{ds} = \frac{dP}{dt}\underbrace{\frac{dt}{ds}}_{=\frac{1}{\dot s}} = \frac{\dot P}{\dot s} = \frac{\dot P}{\|\dot P\|} = \vec{t}$$ versor tangente

$$\boxed{\vec{P}' = \vec{t}}$$

$$\vec{t} = (\cos\alpha, \operatorname{sen}\alpha) \qquad \vec{n} = (-\operatorname{sen}\alpha, \cos\alpha)$$

$$\vec{P}'' = \frac{d\vec{t}}{ds} = \frac{d\vec{t}}{dt}\cdot\underbrace{\frac{dt}{ds}}_{\frac{1}{\dot s}} = \left((-\operatorname{sen}\alpha)\dot\alpha, (\cos\alpha)\dot\alpha\right)\frac{1}{\dot s} =$$

radio de curvatura

$$\vec{p}'' = \frac{d\vec{t}}{ds} = \left(-(\operatorname{sen}\alpha)\dot{\alpha},\ (\cos\alpha)\dot{\alpha}\right)\frac{1}{\dot{s}}$$

$$= \underbrace{\frac{\dot{\alpha}}{\dot{s}}}_{=k}\ \underbrace{(-\operatorname{sen}\alpha, \cos\alpha)}_{\vec{n}}$$

$$\boxed{\vec{p}'' = \frac{d\vec{t}}{ds} = k\cdot\vec{n}} \Rightarrow k \text{ es INTRÍNSECO}$$

DEF: Si $k(t) \neq 0$, "Radio de curvatura" es $R(t) = 1/k(t)$

"Centro de curvatura" punto C sobre la recta normal en $p(t)$, a distancia $|R|$ de p, izq si $R>0$, der si $R<0$

cálculo de la curvatura

CALCULO de CURVATURA   curva de clase C^2

$$\boxed{k(t) = \frac{\dot\alpha(t)}{\dot s(t)}} \qquad \dot s(t) = (\dot x^2 + \dot y^2)^{1/2}$$

Pto regular $(\dot x, \dot y) \neq \vec{0}$

$$\vec{t} = (\cos\alpha, \operatorname{sen}\alpha) = \frac{(\dot x, \dot y)}{(\dot x^2 + \dot y^2)^{1/2}}$$

$$\boxed{\operatorname{tg}\alpha = \frac{\dot y}{\dot x}}$$

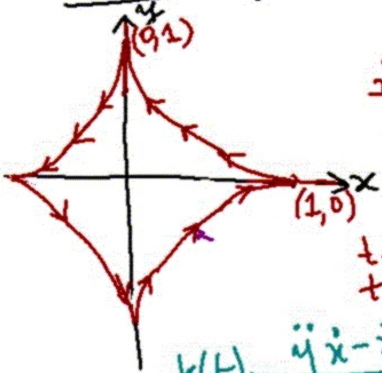
$$\dot\alpha(1 + \operatorname{tg}^2\alpha) = \frac{\ddot y \dot x - \dot y \ddot x}{\dot x^2}$$

$$\dot\alpha\left(\frac{\dot x^2 + \dot y^2}{\dot x^2}\right) \;\Rightarrow\; \dot\alpha(t) = \frac{\ddot y \dot x - \dot y \ddot x}{\dot x^2 + \dot y^2}$$

$$k(t) = \frac{\ddot y \dot x - \dot y \ddot x}{(\dot x^2 + \dot y^2)^{3/2}}$$

ejemplo - la astroide

Ejemplo Dibujar la astroide $\begin{cases} x=\cos^3 t \\ y=\operatorname{sen}^3 t \end{cases}$ $0\le t\le 2\pi$



$\dot{x} = -3\cos^2 t \operatorname{sen} t$

$\dot{y} = 3\operatorname{sen}^2 t \cos t$

$\operatorname{tg}\alpha(t) = \dfrac{\dot{y}}{\dot{x}} = -\operatorname{tg} t$

$t\to 0 \Rightarrow \alpha(t)\to 0$ tgte horizont.

$t\to \pi/2$ $\operatorname{tg}\alpha\to -\infty$ tgte vertical

$$k(t) = \frac{\ddot{y}\dot{x}-\ddot{x}\dot{y}}{\underbrace{(\dot{x}^2+\dot{y}^2)^{3/2}}_{\geq 0}} = \frac{-9\cos^2 t \operatorname{sen}^2 t}{(\dot{x}^2+\dot{y}^2)^{3/2}} < 0$$

CONCAVIDAD HACIA LA DERECHA

astroide - curvatura