Integral curvilínea
Uno-forma

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Cálculo 3

IMERL

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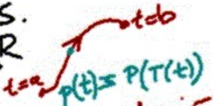
[1] Sobre notas de Eleonora Catsigeras

obs - integral no depende de parametrización

CAMBIO de PARÁMETROS EN INTEGRALES.



Sea $A(x,y,z)$ función continua $A : \underset{\text{abierto de } \mathbb{R}^3}{\Omega} \longrightarrow \mathbb{R}$

$\mathcal{C}$ curva paramétrica $\mathcal{C} \subset \Omega$ orientada

$x(t), y(t), z(t)$ $\quad a \le t \le b$ una parametriz de $\mathcal{C}$ } con la orientación

$X(T), Y(T), Z(T)$ $\quad c \le T \le d$ otra " " $\mathcal{C}$ } = para $t\uparrow$ y para $T\uparrow$

Es decir: $\exists$ cambio de parámetros $T = T(t)$ $\quad T(a) = c \quad T(b) = d$

$$\left(\boxed{x(t) = X(T(t))};\ y(t) = Y(T(t)),\ z(t) = Z(T(t)) \right)$$

$\boxed{\text{TEOREMA}}$

$$I = \underbrace{\int_a^b A(x(t),y(t),z(t))\,\dot{x}(t)\,dt}_{I_1} = \underbrace{\int_c^d A(X(T),Y(T),Z(T))\,\dot{X}(T)\,dT}_{I_2}$$

Dem. En I_2 cambio de variables $T = T(t)$

$$\int_a^b A(\underbrace{X(T(t))}_{x(t)}, \underbrace{Y(T(t))}_{y(t)}, \dots)\ \underbrace{\dot{X}(T(t))\,\dot{T}(t)}_{\dot{x}(t)}\,dt$$

$\Rightarrow I_2 = I_1$ Conclusión: I depende sólo de $\mathcal{C}$ orientada y de $A(x,y,z)$

I se denota $\displaystyle\int_{\mathcal{C}} A(x,y,z)\,dx$

uno-forma e integral curvilínea

$$\int_{\mathcal{C}} A(x,y,z)\,dx = \int_a^b A(x(t),y(t),z(t))\,\dot{x}(t)\,dt$$

Análogamente $\displaystyle\int_{\mathcal{C}} B(x,y,z)\,dy = \int_a^b B(x(t),y(t),z(t))\,\dot{y}(t)\,dt$

$$\int_{\mathcal{C}} C(x,y,z)\,dz = \int_a^b C(x(t)\ldots\ \cdot)\,\dot{z}(t)\,dt$$

no dependen de la parametrización elegida.

La suma de las tres integrales tampoco.

DEFINICIÓN $A(x,y,z)\,dx + B(x,y,z)\,dz + C(x,y,z)\,dz$ es UNO-FORMA DIFERENCIAL o FORMA DIF. LINEAL

$$\int_{\mathcal{C}} A\,dx + B\,dy + C\,dz \stackrel{\text{def}}{=} \int_a^b A(x(t),y(t),z(t))\,\dot{x}(t) + B\,\dot{y} + C\,\dot{z}\ \ dt$$

es "INTEGRAL CURVILÍNEA de la UNO-FORMA $A\,dx + B\,dy + C\,dz$ a lo largo de $\mathcal{C}$"

Ejemplos

$\mathcal{C}$ cfa. de centro $(0,0)$ radio 1, sent antihor

$$L = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \quad \text{en } \Omega = \mathbb{R}^2 \setminus \{(0,0)\}$$

Calcular $\int_{\mathcal{C}} L$

$$\mathcal{C}: \begin{cases} x = \cos t \\ y = \operatorname{sen} t \end{cases} \quad 0 \le t \le 2\pi$$

$$I = \int_0^{2\pi} \frac{\overbrace{-\operatorname{sen} t}^{-y(t)}}{1} \overbrace{(-\operatorname{sen} t)}^{\dot{x}(t)} + \frac{\overbrace{\cos t}^{x(t)}}{1} \overbrace{\cos t}^{\dot{y}(t)}\,dt$$

$$I = \int_0^{2\pi} \underbrace{(\operatorname{sen}^2 t + \cos^2 t)}_{1}\,dt = 2\pi$$

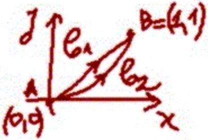
$\mathcal{C}$ cerrada

$\int_{\mathcal{C}} L \neq 0$ en este ejemplo

ejemplo 2

<u>Ejemplo</u> $\mathcal{C}_1=\{y=x : 0\le x\le 1\}$ $x\uparrow$

$L = y\,dx$ $\quad \mathcal{C}_2=\{y=x^2 : 0\le x\le 1\}$ $x\uparrow$



Calcular $\int_{\mathcal{C}_1} L$ y $\int_{\mathcal{C}_2} L$

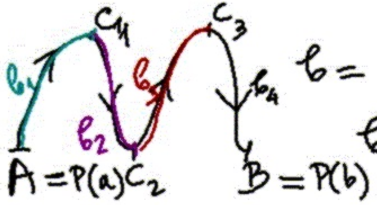
$$\mathcal{C}_1: \begin{cases} x=t \\ y=t \end{cases} \quad 0\le t\le 1 \quad \int_{\mathcal{C}_1} L = \int_0^1 y(t)\,\dot{x}(t)\,dt = \int_0^1 t\,dt = \frac{1}{2}$$

$$\mathcal{C}_2: \begin{cases} x=t \\ y=t^2 \end{cases} \quad 0\le t\le 1 \quad \int_{\mathcal{C}_2} L = \int_0^1 y(t)\,\dot{x}(t)\,dt = \int_0^1 \boxed{t^2}\,dt = \frac{1}{3}$$

bien

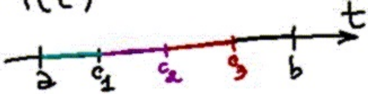
$I_1 \neq I_2$ La integral de L cambia en general al modificar el recorrido entre A y B

suma de integrales curvilíneas



$$\mathcal{C} = \mathcal{C}_1 \dotplus \mathcal{C}_2 \dotplus \mathcal{C}_3 \dotplus \mathcal{C}_4$$

$$\mathcal{C} : P(t) \qquad a \leq t \leq b$$



$$\int_{\mathcal{C}} A\,dx = \int_a^b A\dot{x}\,dt = \underbrace{\int_a^{c_1}}_{\int_{\mathcal{C}_1} A\,dx} + \underbrace{\int_{c_1}^{c_2}}_{\int_{\mathcal{C}_2} A\,dx} + \underbrace{\int_{c_2}^{c_3}}_{\int_{\mathcal{C}_3} A\,dx} + \underbrace{\int_{c_3}^{b}}_{\int_{\mathcal{C}_4} A\,dx} =$$

$$\int_{\mathcal{C}_1 \dotplus \mathcal{C}_2 \dotplus \mathcal{C}_3 \dotplus \mathcal{C}_4} L = \int_{\mathcal{C}_1} L + \int_{\mathcal{C}_2} L + \int_{\mathcal{C}_3} L + \int_{\mathcal{C}_4} L$$

cambio de orientación

$\mathcal{C}$ orientada

$P=P(t);\ a\le t\le b$ $\quad A=P(a)$, $\quad B=P(b)$, $\quad \mathcal{C}$

$-\mathcal{C}$ denota la misma curva con la orientación opuesta

$-\mathcal{C}$

$P=P(-T)\quad -b\le T\le -a$

$X(T)=x(-T)$ $\quad A=P(-(-a))$ $\quad B=P(-(-b))$

$\boxed{\text{TEOREMA}}$

$$\int_{-\mathcal{C}} L = -\int_{\mathcal{C}} L$$

Dem

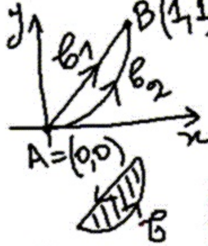
$$\int_{-\mathcal{C}} A\,dx = \int_{-b}^{-a} A(X(T),\dots)\,\dot X(T)\,dT = \int_b^a A(x(t),\dots)\,(-\dot x(t))(-dt) = -\int_a^b A(x(t)\dots)\,\dot x\,dt = -\int_{\mathcal{C}} A\,dx$$

$X(T) = x(-T)$, $\quad \dot X(T) = -\dot x(-T)$

cambio $T=-t$, $\quad dT=-dt$

ejemplo - cálculo de área

Ejemplo



B(1,1)

$$\mathcal{C}_1 = \{y = x;\ 0 \le x \le 1\}\ x\uparrow$$

$$\mathcal{C}_2 = \{y = x^2;\ 0 \le x \le 1\}\ x\uparrow$$

$$\int_{\mathcal{C}_1} y\,dx = \frac{1}{2} \qquad \int_{\mathcal{C}_2} y\,dx = \frac{1}{3}$$

$$\mathcal{C} = \mathcal{C}_2 \cup (-\mathcal{C}_1)$$

$$\text{Área} = -\int_{\mathcal{C}} y\,dx = -\left(\int_{\mathcal{C}_2} y\,dx + \int_{-\mathcal{C}_1} y\,dx\right)$$

$$= -\left(\underbrace{\int_{\mathcal{C}_2} y\,dx}_{1/3} - \underbrace{\int_{\mathcal{C}_1} y\,dx}_{1/2}\right) = -\left(\frac{1}{3} - \frac{1}{2}\right) = \frac{1}{6}$$