Campos conservativos y formas exactas

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Cálculo 3

IMERL

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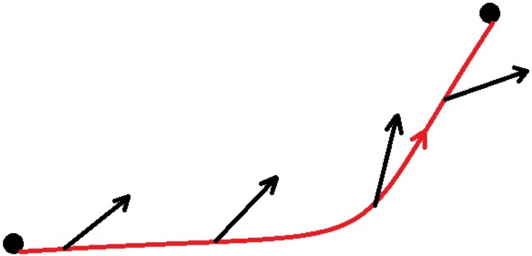
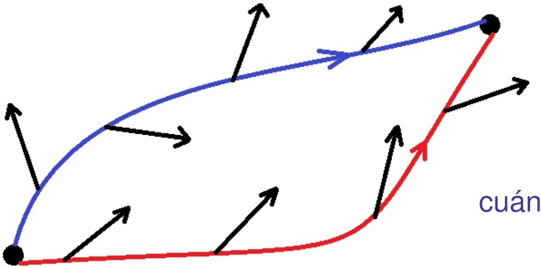
motivación



$$\int_{\mathcal{C}_1} \vec{X}\,\vec{ds}$$

motivación



$$\int_{\mathcal{C}_1} \vec{X}\overrightarrow{ds} \stackrel{?}{=} \int_{\mathcal{C}_2} \vec{X}\overrightarrow{ds}$$

cuándo?

teorema

teorema (campo conservativo)

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- $\Omega \subset \mathbb{R}^3$ conexo

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- $\vec{X} : \Omega \to \mathbb{R}^3$ campo vectorial

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- Son equivalentes:

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  1. $\vec{X} = \nabla f$

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- Son equivalentes:
  1. $\vec{X} = \nabla f$
  2. $\int_{\mathcal{C}} \vec{X} \overrightarrow{ds} = 0 \qquad \forall \mathcal{C} \subset \Omega$ cerrada

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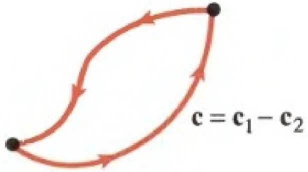
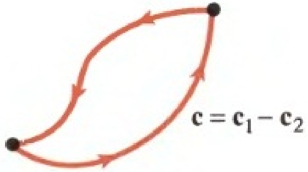
- $\Omega \subset \mathbb{R}^3$ conexo
- $\vec{X} : \Omega \to \mathbb{R}^3$ campo vectorial
- Son equivalentes:
  1. $\vec{X} = \nabla f$
  2. $\int_{\mathcal{C}} \vec{X}\overrightarrow{ds} = 0 \qquad \forall \mathcal{C} \subset \Omega$ cerrada
  3. $\int_{\mathcal{C}_1} \vec{X}\overrightarrow{ds} = \int_{\mathcal{C}_2} \vec{X}\overrightarrow{ds} \qquad \forall \mathcal{C}_1, \mathcal{C}_2 \subset \Omega$ orientadas entre P y Q

dem 2 $\Leftrightarrow$ 3



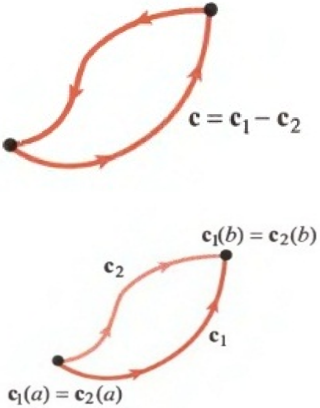
$$\int_C \vec{X}\,\overrightarrow{ds} = 0$$

dem 2 $\Leftrightarrow$ 3



$$\begin{aligned} \int_{\mathcal{C}} \vec{X}\overrightarrow{ds} &= 0 \\ \int_{\mathcal{C}_1 - \mathcal{C}_2} \vec{X}\overrightarrow{ds} &= 0 \end{aligned}$$

dem 2 $\Leftrightarrow$ 3



$$\int_{\mathcal{C}} \vec{X}\overrightarrow{ds} = 0$$

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$$\int_{\mathcal{C}_1} \vec{X}\overrightarrow{ds} - \int_{\mathcal{C}_1} \vec{X}\overrightarrow{ds} = 0$$

demostración 1 $\Rightarrow$ 3

- $\vec{X} = (f_x, f_y, f_z)$

demostración 1 $\Rightarrow$ 3

- $\vec{X} = (f_x, f_y, f_z)$
- $\mathcal{C}_i : P_i(t) = (x(t), y(t), z(t))$, con $P = P_i(a)$, $Q = P_i(b)$

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- $\mathcal{C}_i : P_i(t) = (x(t), y(t), z(t))$, con $P = P_i(a)$, $Q = P_i(b)$

$$\int_{\mathcal{C}_i} \vec{X}\overrightarrow{ds} \quad = \quad \int_a^b (f_x, f_y, f_z)(\dot{x}, \dot{y}, \dot{z})dt$$

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$$\begin{aligned}
\int_{\mathcal{C}_i} \vec{X}\overrightarrow{ds} &= \int_a^b (f_x, f_y, f_z)(\dot{x}, \dot{y}, \dot{z})dt \\
&= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt
\end{aligned}$$

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&= \int_a^b d(f \circ P_i)
\end{aligned}$$

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$\Rightarrow \int_{\mathcal{C}_1} \vec{X}\overrightarrow{ds} = \int_{\mathcal{C}_2} \vec{X}\overrightarrow{ds}$

observación importante

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- $\int_{\mathcal{C}} \vec{X} \overrightarrow{ds} \neq 0$

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- $\mathcal{C} \subset \Omega$ curva cerrada
- $\int_{\mathcal{C}} \vec{X} \overrightarrow{ds} \neq 0$
- $\Rightarrow \vec{X}$ no es de gradientes en Ω

ejemplo

- $\vec{X} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$ en $\Omega = \mathbb{R}^2 \setminus (0,0)$

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- ya vimos que $\int_{\mathcal{C}} \vec{X}\overrightarrow{ds} = 2\pi \neq 0$
- $\Rightarrow \vec{X}$ no es conservativo en Ω (no tiene potencial)
- $\Rightarrow L_X = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ no es exacta

dem 3 $\Rightarrow$ 1

- Suponer $\int_{\mathcal{C}_1} \vec{X}\overrightarrow{ds} = \int_{\mathcal{C}_2} \vec{X}\overrightarrow{ds}$

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- Fijar $P_0 = (x_0, y_0, z_0)$
- Definir $f(x, y, z) = \int_{\mathcal{C}(P_0, P)} \vec{X}\overrightarrow{ds}$

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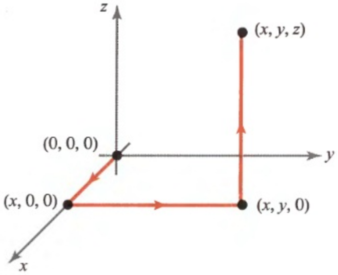
- 

dem 3 $\Rightarrow$ 1

$$f_x(x,y,z) = \lim_{\triangle x \to 0} \frac{f(x+\triangle x, y, z) - f(x,y,z)}{\triangle x}$$

dem 3 $\Rightarrow$ 1

$$
\begin{aligned}
f_x(x,y,z) &= \lim_{\triangle x \to 0} \frac{f(x+\triangle x, y, z) - f(x,y,z)}{\triangle x} \\
&= \lim_{\triangle x \to 0} \frac{1}{\triangle x} \left(\int_{\mathcal{C}(P_0, P+\triangle x)} \vec{X} \overrightarrow{ds} - \int_{\mathcal{C}(P_0, P)} \vec{X} \overrightarrow{ds} \right)
\end{aligned}
$$

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&= \lim_{\triangle x \to 0} \frac{1}{\triangle x} \int_{\mathcal{C}(P, P+\triangle x)} \vec{X}\,\overrightarrow{ds}
\end{aligned}
$$

encontrando un potencial

dem 3 ⇒ 1

$$\begin{aligned} f_x(x,y,z) &= \lim_{\triangle x \to 0} \frac{f(x+\triangle x, y, z) - f(x,y,z)}{\triangle x} \\ &= \lim_{\triangle x \to 0} \frac{1}{\triangle x} \left(\int_{\mathcal{C}(P_0, P+\triangle x)} \vec{X}\overrightarrow{ds} - \int_{\mathcal{C}(P_0,P)} \vec{X}\overrightarrow{ds} \right) \\ &= \lim_{\triangle x \to 0} \frac{1}{\triangle x} \int_{\mathcal{C}(P, P+\triangle x)} \vec{X}\overrightarrow{ds} \\ &= \lim_{\triangle x \to 0} \frac{1}{\triangle x} \int_0^{\triangle x} A(x+t, y, z)dt \end{aligned}$$

dem 3 $\Rightarrow$ 1

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&= A(x,y,z)
\end{aligned}
$$

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  2. $\int L_{\mathcal{C}} L = 0 \qquad \forall \mathcal{C} \subset \Omega$ cerrada

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  3. $\int_{\mathcal{C}_1} L = \int_{\mathcal{C}_2} L \qquad \forall \mathcal{C}_1, \mathcal{C}_2 \subset \Omega$ orientadas entre P y Q