

The use of blenders in smooth ergodic theory

A case study

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R. Ures

question

How frequent is ergodicity?

stable ergodicity

$f : M \rightarrow M$ is stably ergodic if there is $\mathcal{U} \subset C^1$ open s.t.

stable ergodicity

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$$f \in \mathcal{U} \cap \text{Diff}_m^2(M) \subset \text{ergodic}$$

question

sufficient conditions for stable ergodicity?

partial hyperbolicity

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$$TM = E^s \oplus E^c \oplus E^u$$

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\uparrow contracting \uparrow expanding

C^r ergodic Pugh-Shub conjecture (1995)

Stable ergodicity is C^r dense among partially hyperbolic diffeomorphisms.

Theorem - R.Hertz, R.H. Tahzibi, Ures (2007)

Stable ergodicity is C^1 dense among C^2 partially hyperbolic diffeomorphisms with $\dim E^c = 2$.

elements of the proof

- ▶ stable accessibility is C^1 -dense among PH-diffeomorphisms (DW)

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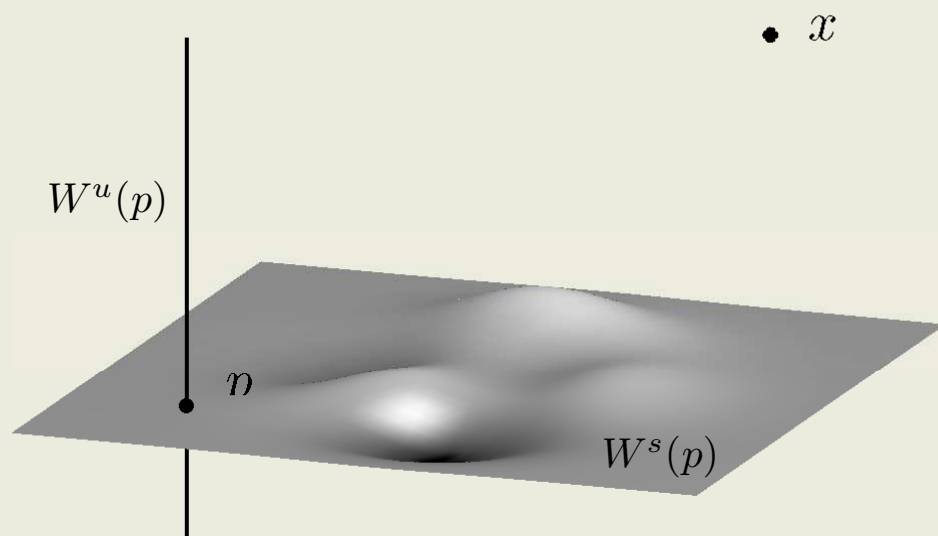
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- ▶ New description of ergodic components
- ▶ Use of blenders

ergodic homoclinic classes

$x \in \Lambda(p)$ $p \in \text{Per}(f)$ hyperbolic



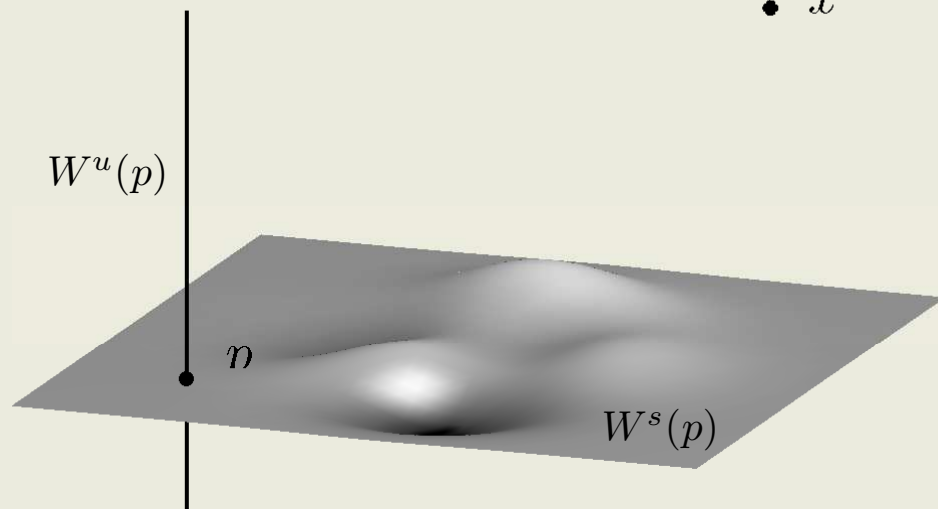
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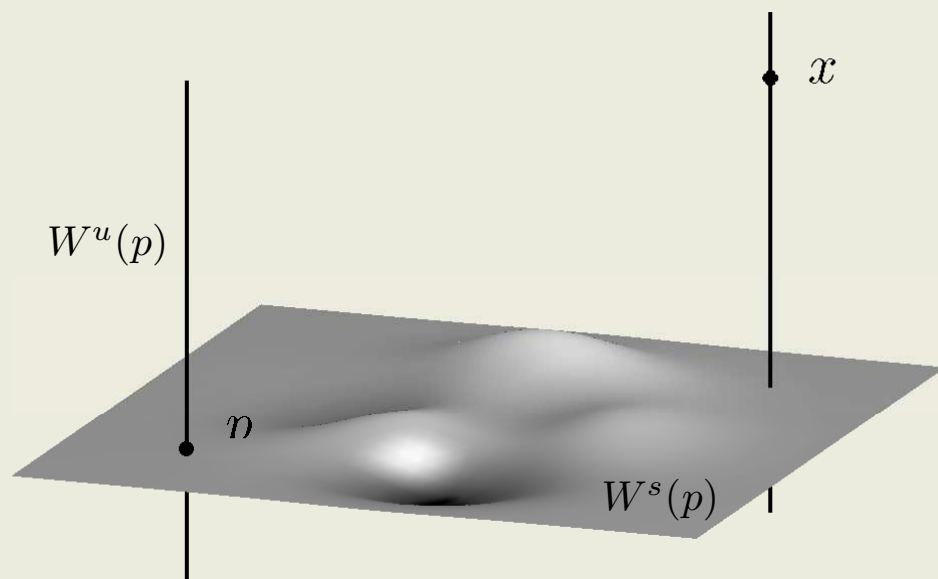
x regular point

• x



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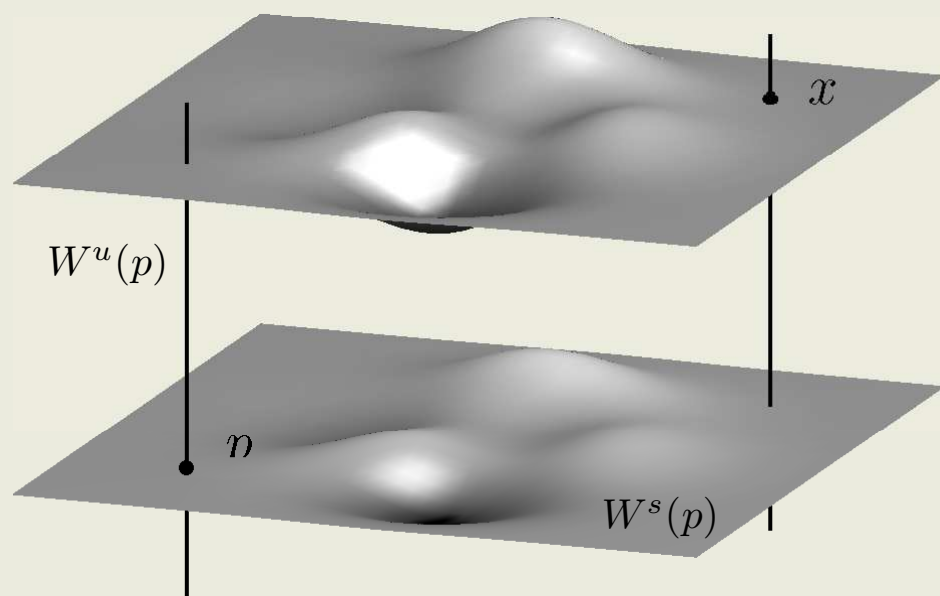


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$$W^u(x) \cap W^s(o(p)) \neq \emptyset$$

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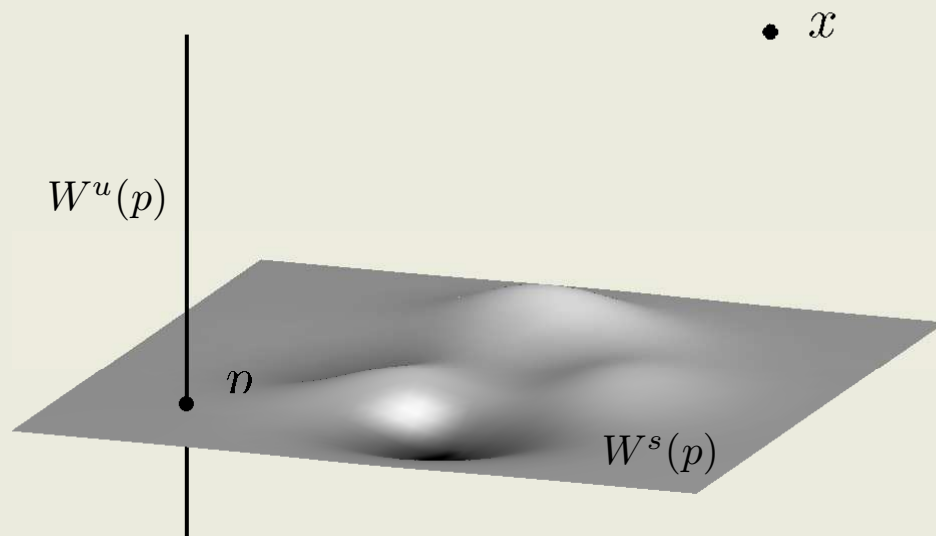
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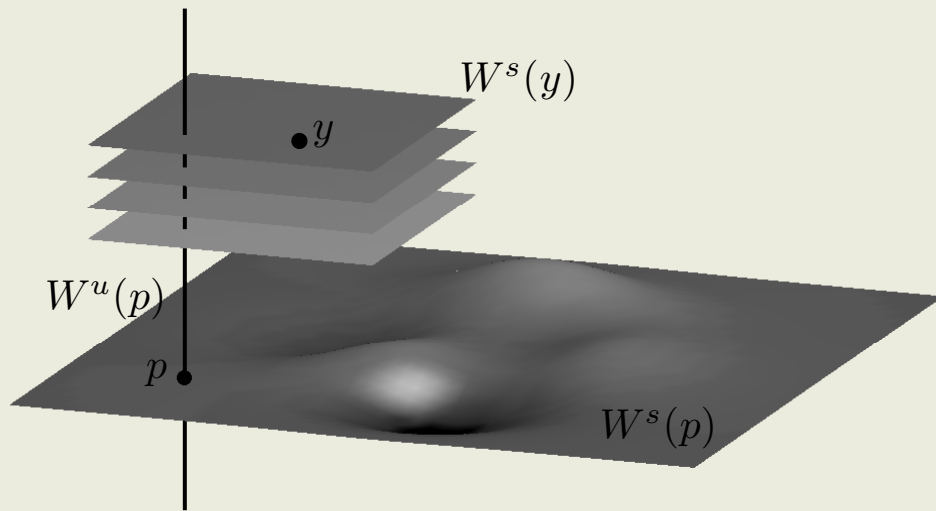
$\Lambda^s(p)$ and $\Lambda^u(p)$

$$\Lambda(p) = \Lambda^s(p) \cap \Lambda^u(p) \quad \text{where}$$



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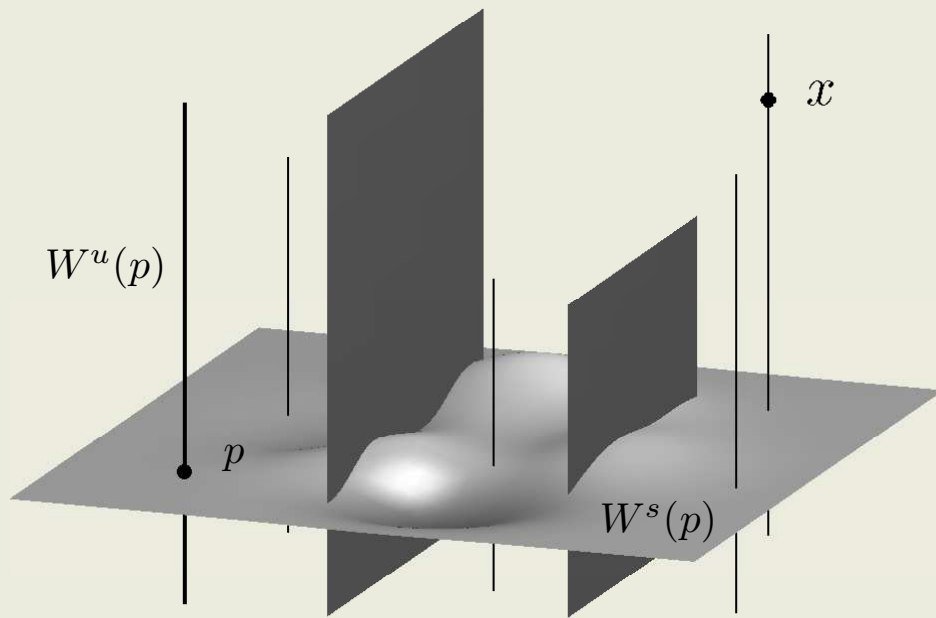
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description of ergodic components

$f \in \text{Diff}^{1+\alpha}(M)$ and m a smooth measure.

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$f \in \text{Diff}^{1+\alpha}(M)$ and m a smooth measure. If

$$m(\Lambda^s(p)) > 0 \quad \text{and} \quad m(\Lambda^u(p)) > 0$$

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► $\Lambda(p) \stackrel{\circ}{=} \Lambda^s(p) \stackrel{\circ}{=} \Lambda^u(p)$

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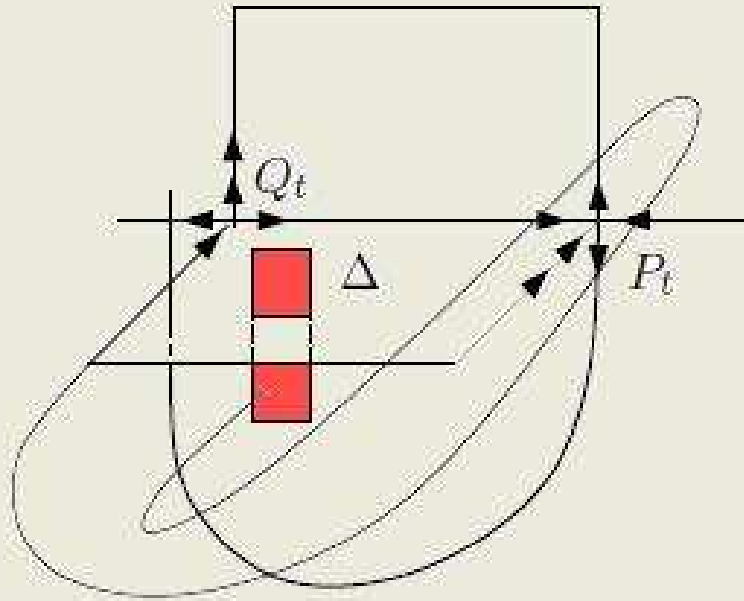
- ▶ $\Lambda(p) \stackrel{\circ}{=} \Lambda^s(p) \stackrel{\circ}{=} \Lambda^u(p)$
- ▶ $f|_{\Lambda(p)}$ ergodic

p homoclinically related to q



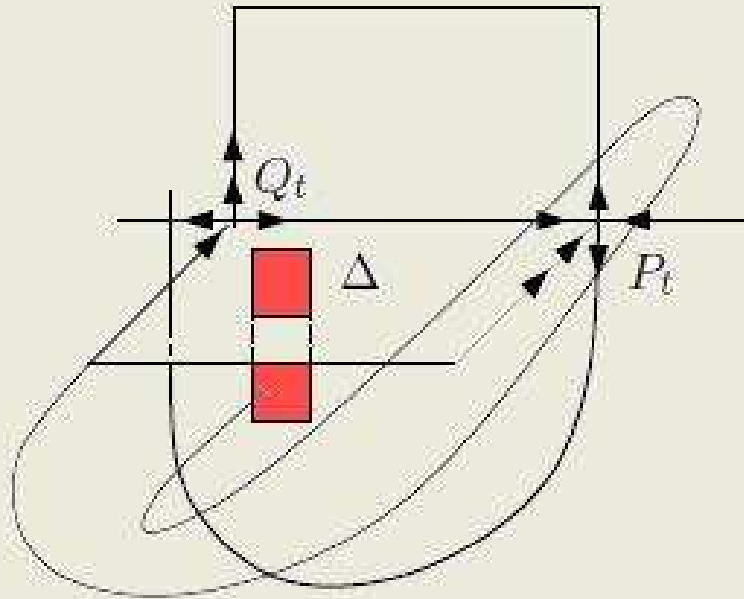
$$\Lambda(p) \stackrel{\circ}{=} \Lambda(q)$$

what is a blender?



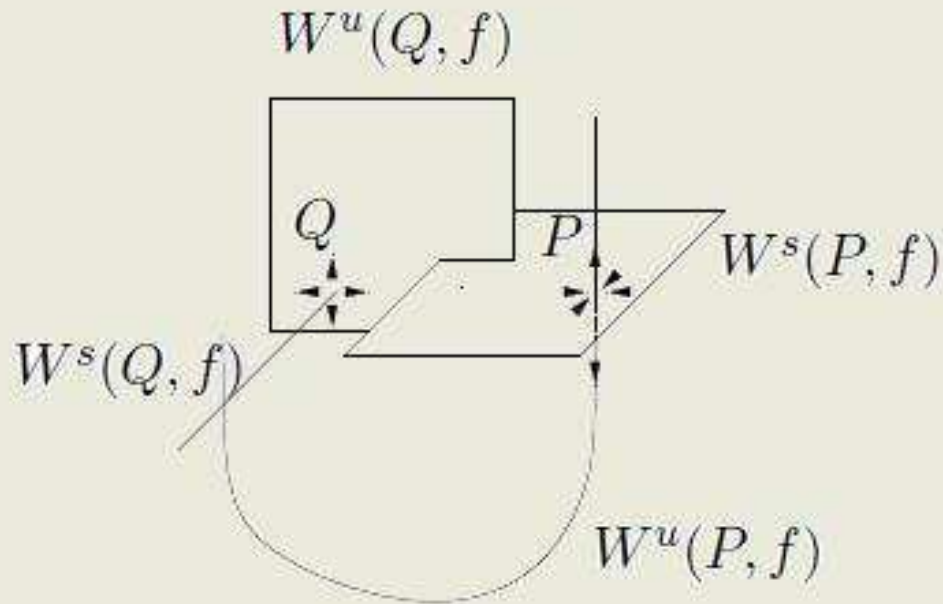
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what is a blender?



dynamics generated by
a heterodimensional cycle

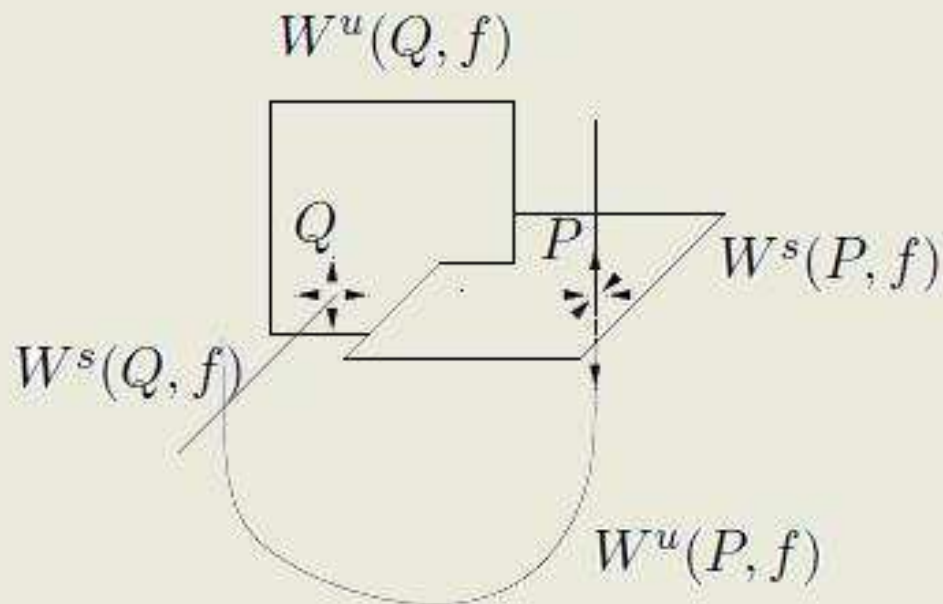
heterodimensional cycle



► $P, Q \in \text{Per}_H(f)$

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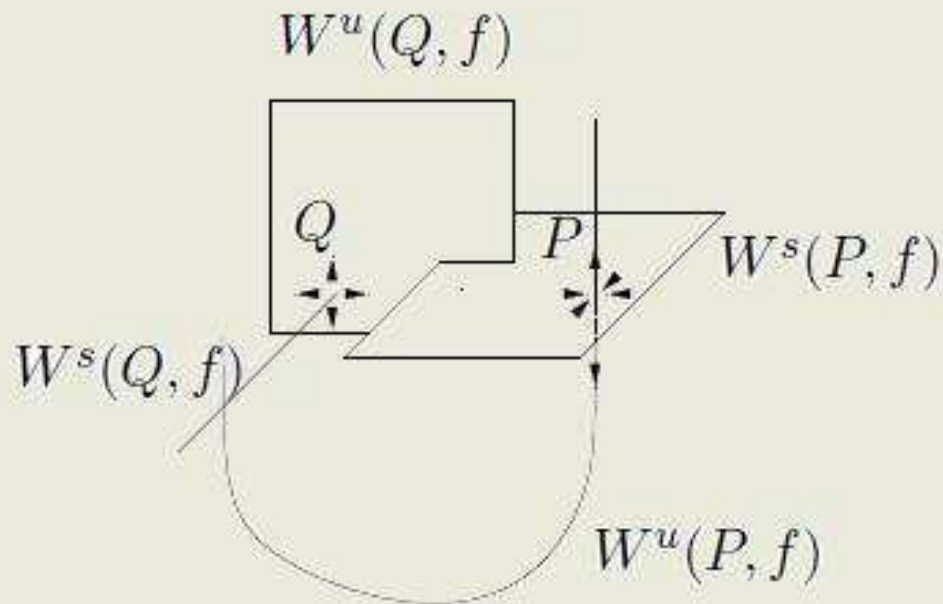
heterodimensional cycle



- ▶ $P, Q \in \text{Per}_H(f)$
- ▶ $\text{ind}_s(P) = \text{ind}_s(Q) + 1$

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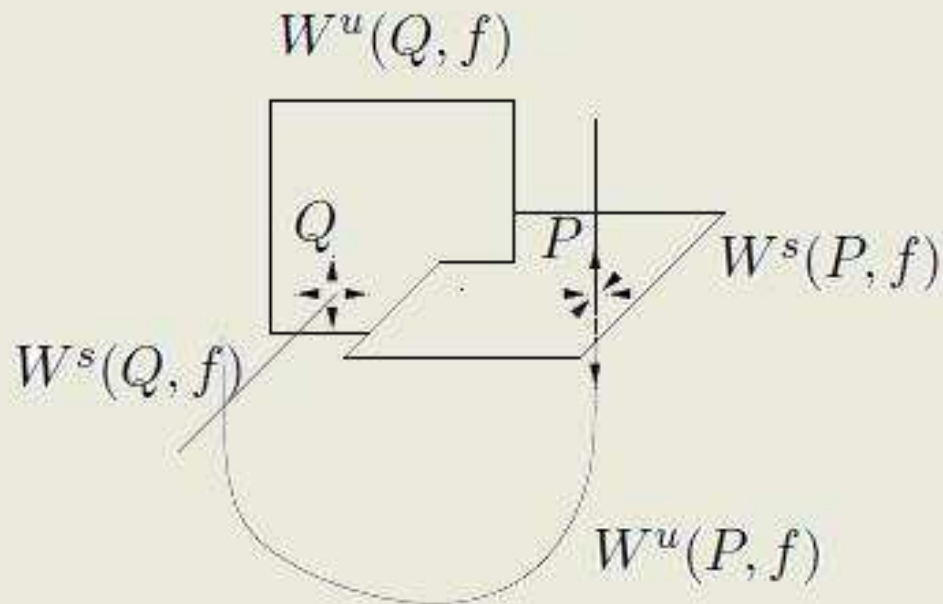
heterodimensional cycle



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- ▶ $W^s(P) \pitchfork W^u(Q)$

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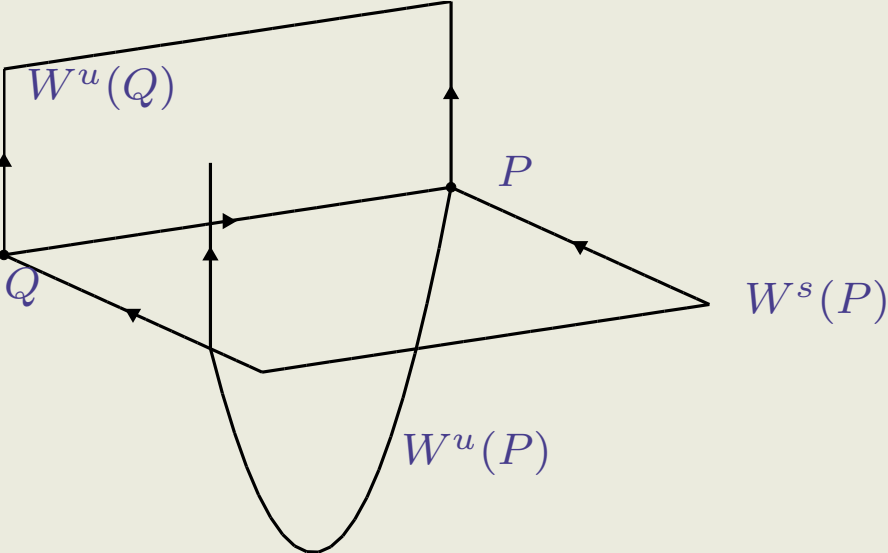
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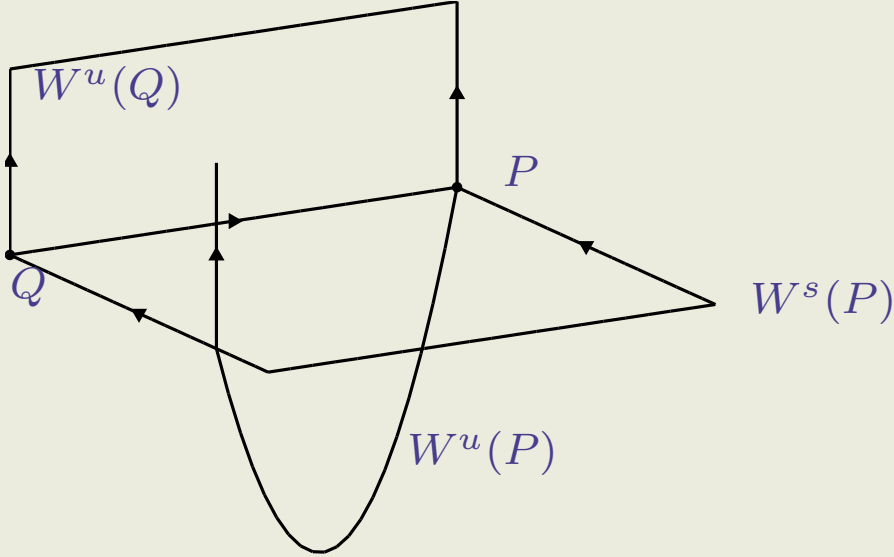
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creation of a blender

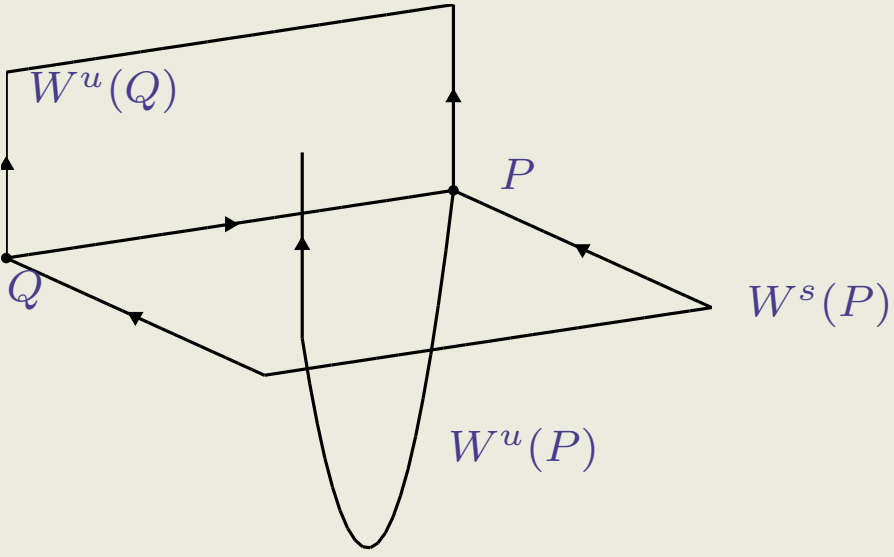


before

creation of a blender

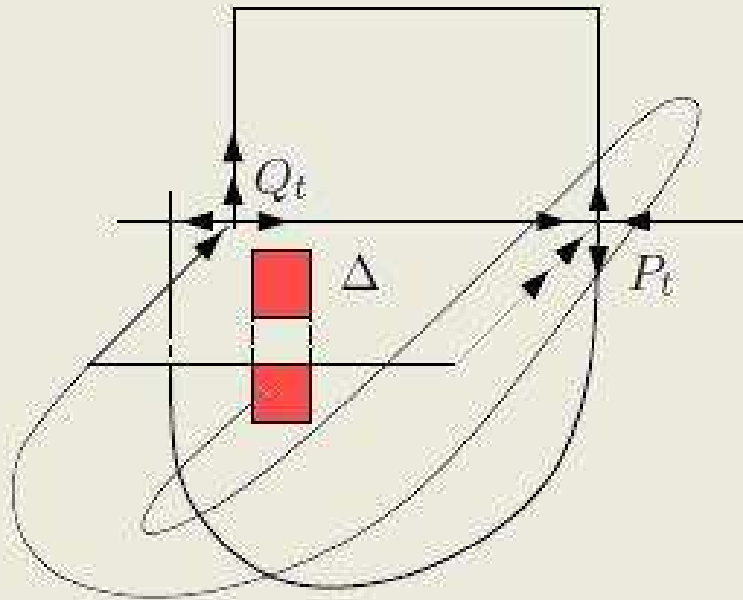


before



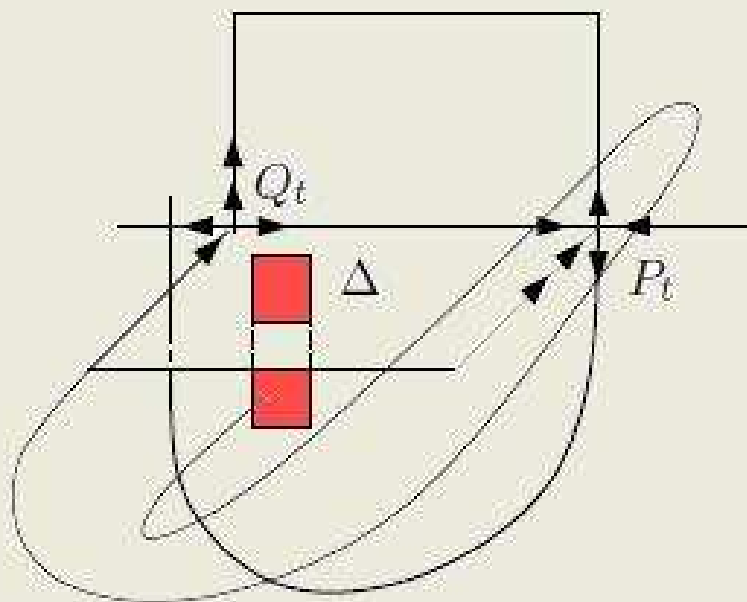
after

properties of blenders



- C^1 -near a heterodimensional cycle

properties of blenders



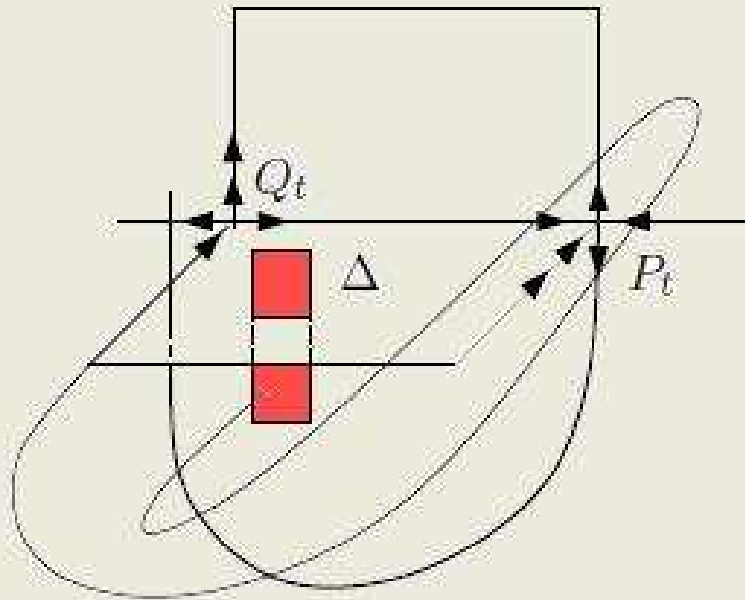
► C^1 -near a heterodimensional cycle

► Δ^{u+1} well placed

⇓

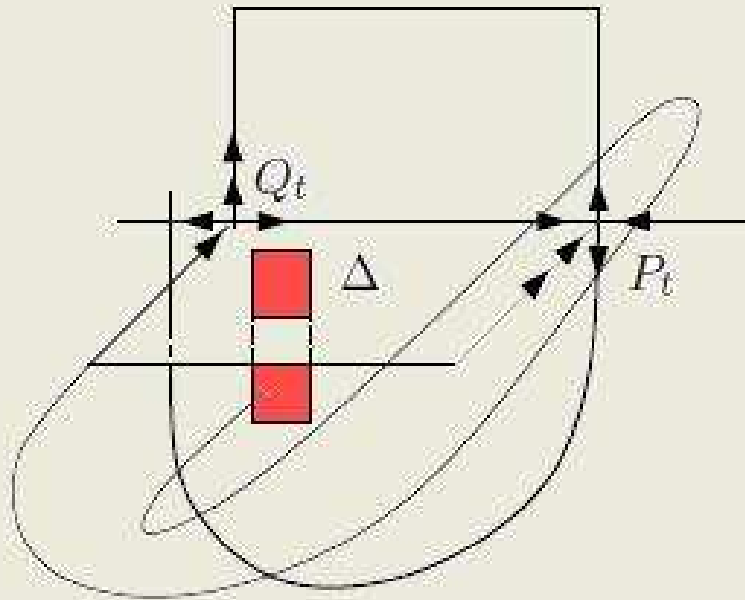
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sketch of the proof

for $f \in C^2$ partially hyperbolic with $\dim E^c = 2$, either

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$\lambda_g^+(x) \geq \lambda_g^-(x)$ Lyapunov exponents of $g|_{E^c}$

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- $\Rightarrow f$ is stably ergodic

step ①

① g stably accessible

Stable accessibility is dense (DW)

step ②

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Bochi-Viana type argument

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Bochi-Viana type argument

$$\int_M \lambda_g^- dm \geq \int_M \lambda_f^- dm + \int_{NDS} \frac{\lambda_f^+ - \lambda_f^-}{2} dm - \varepsilon$$

step 2

② $E_g^c = E_g^- \oplus E_g^+$ non-trivial dominated splitting

Bochi-Viana type argument

$$\begin{aligned} \int_M \lambda_g^- dm &\geq \int_M \lambda_f^- dm + \int_{NDS} \frac{\lambda_f^+ - \lambda_f^-}{2} dm - \varepsilon \\ &\geq \int_M \frac{\lambda_f^+ + \lambda_f^-}{2} - \varepsilon \end{aligned}$$

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note

$$\int_M (\lambda_f^+ + \lambda_f^-) dm = \int_M \log \text{Jac}(Tf|_{E_f^c}) dm > 0 \quad (\text{BB})$$

step ③

$$\textcircled{3} \int \lambda_g^- dm < 0 \text{ and } \int \lambda_g^+ dm > 0$$

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Baraviera-Bonatti:

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Baraviera-Bonatti:

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Then either $\int_M \lambda_g^+ dm < 0$ or $\int_M \lambda_g^- dm > 0$

(BDP) applies

step 3

$$\textcircled{3} \int \lambda_g^- dm < 0 \text{ and } \int \lambda_g^+ dm > 0$$

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$$\text{Then either } \int_M \lambda_g^+ dm < 0 \text{ or } \int_M \lambda_g^- dm > 0$$

(BDP) applies

or else $\textcircled{3}$

step 4

4a g has hyperbolic periodic points p_s, p_{s+1}, p_{s+2}

▶ E_f^+ and E_f^- are not hyperbolic

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4a g has hyperbolic periodic points p_s, p_{s+1}, p_{s+2}

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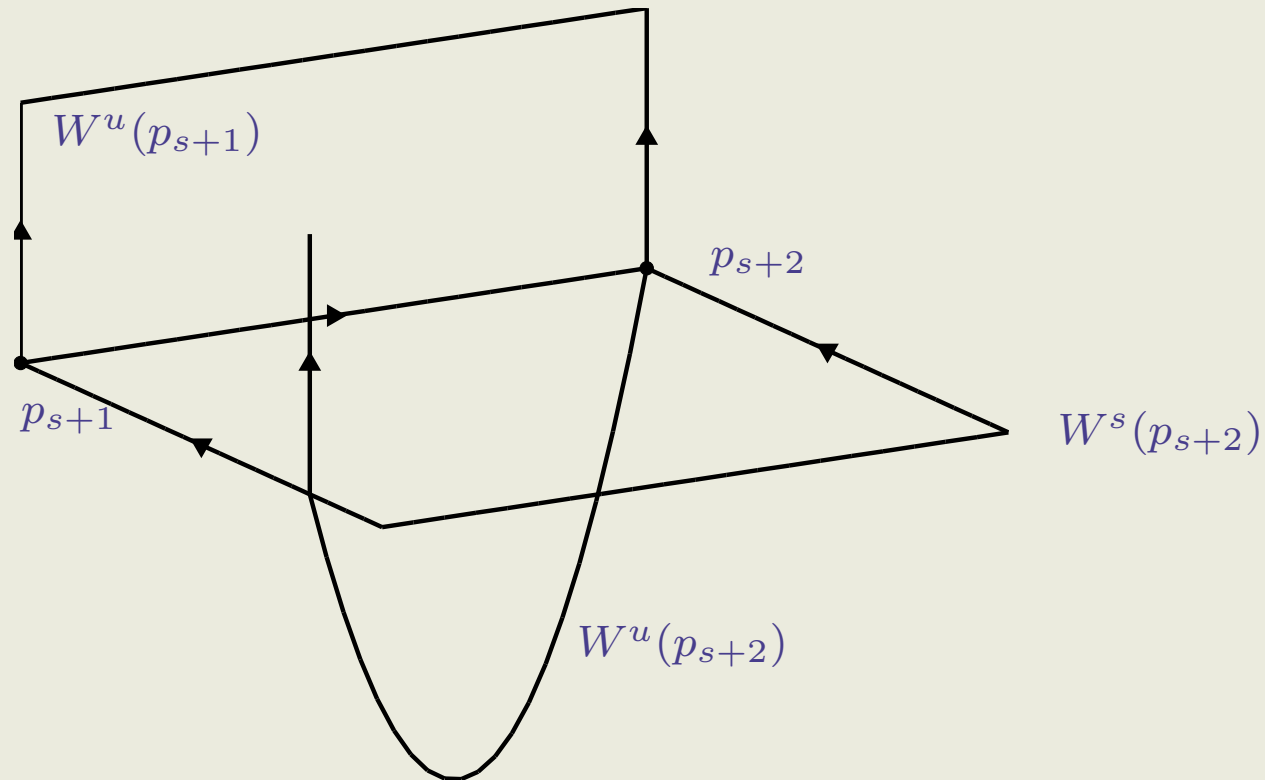
step 4

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- ▶ and $\lambda_g^+(p_{s+1}) \cdot \lambda_g^-(p_{s+1}) < 0$

step 4

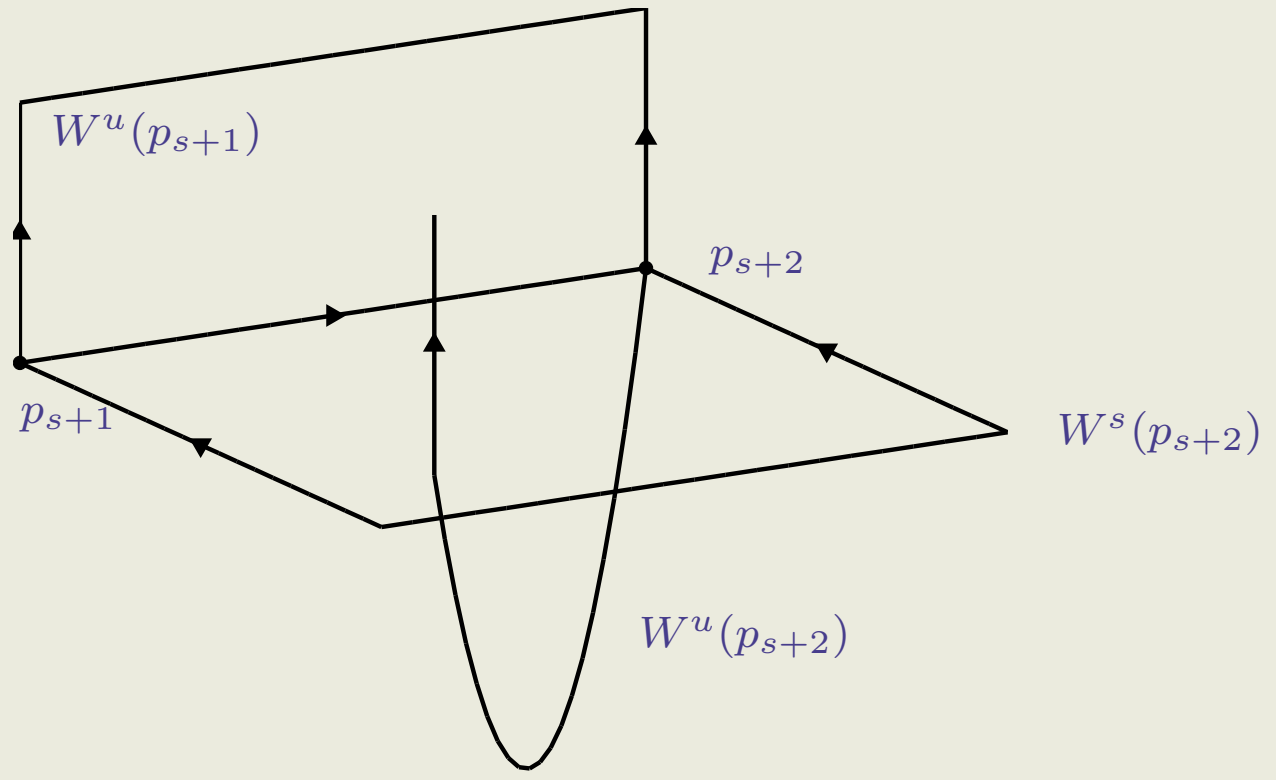
4b $\exists \text{BI}^{cu}(p_{s+1})$



conservative connecting lemma

step 4

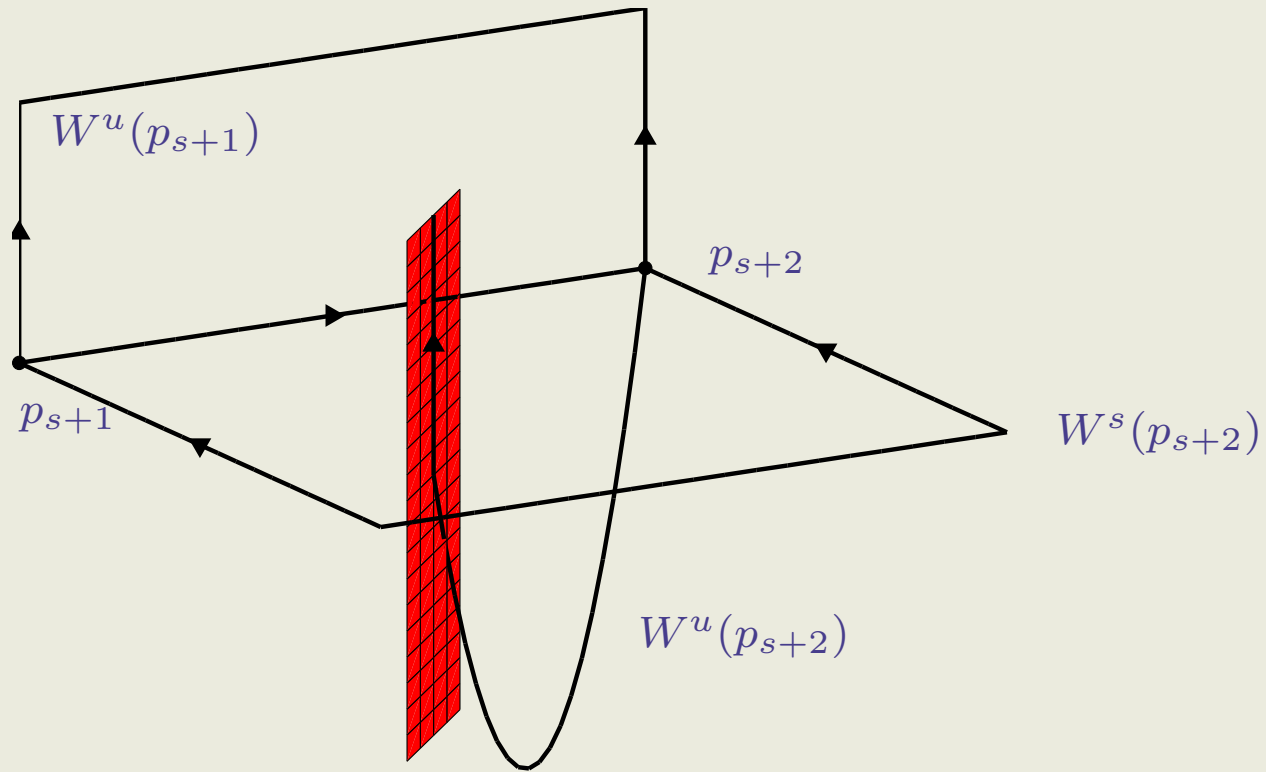
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creation of blender

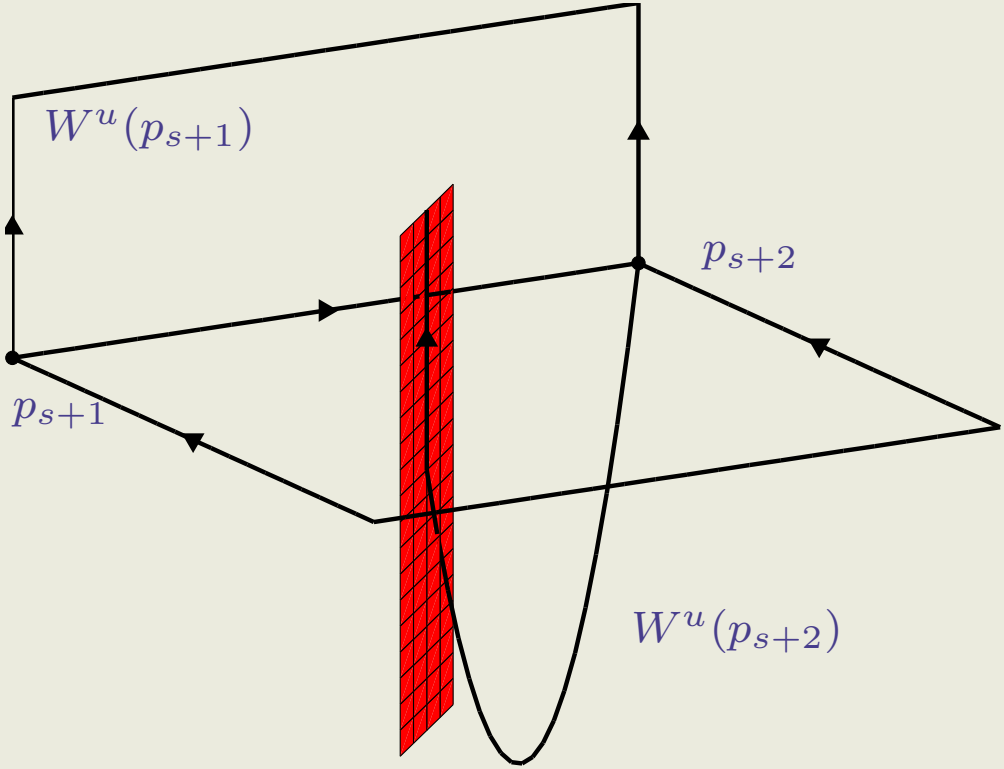
step 4

4b $\exists \text{BI}^{cu}(p_{s+1})$



every vertical strip $\cap W^s(p_{s+1})$

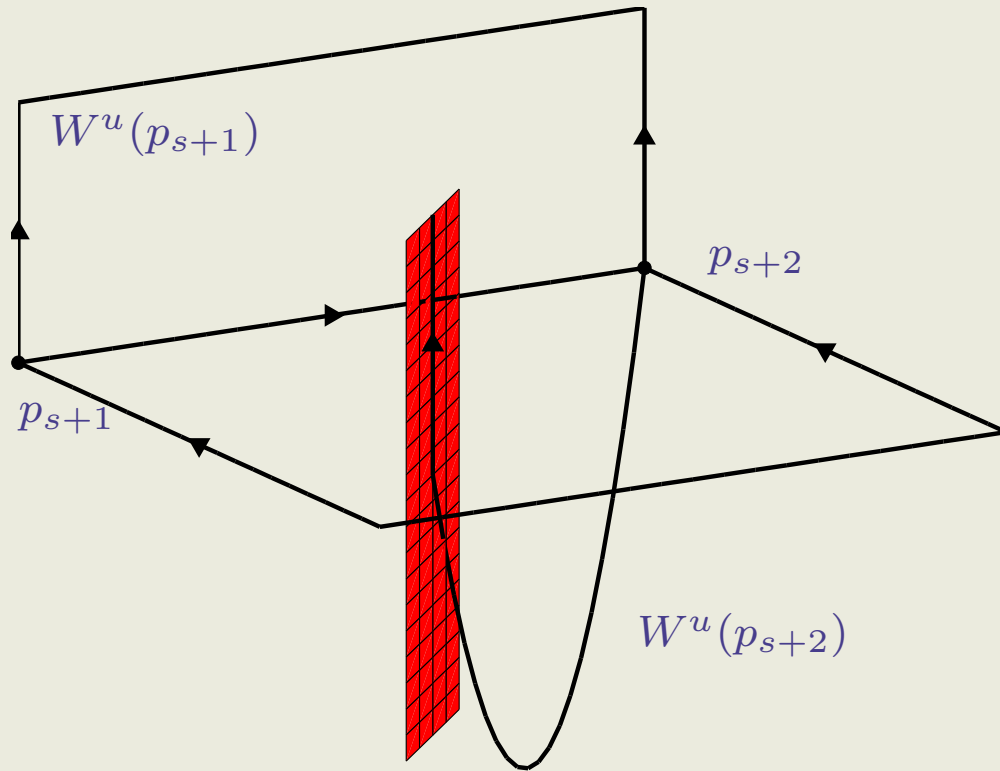
ergodicity



$$C^+ = x : \lambda^+(x) > 0$$

$$C^- = x : \lambda^-(x) < 0$$

ergodicity

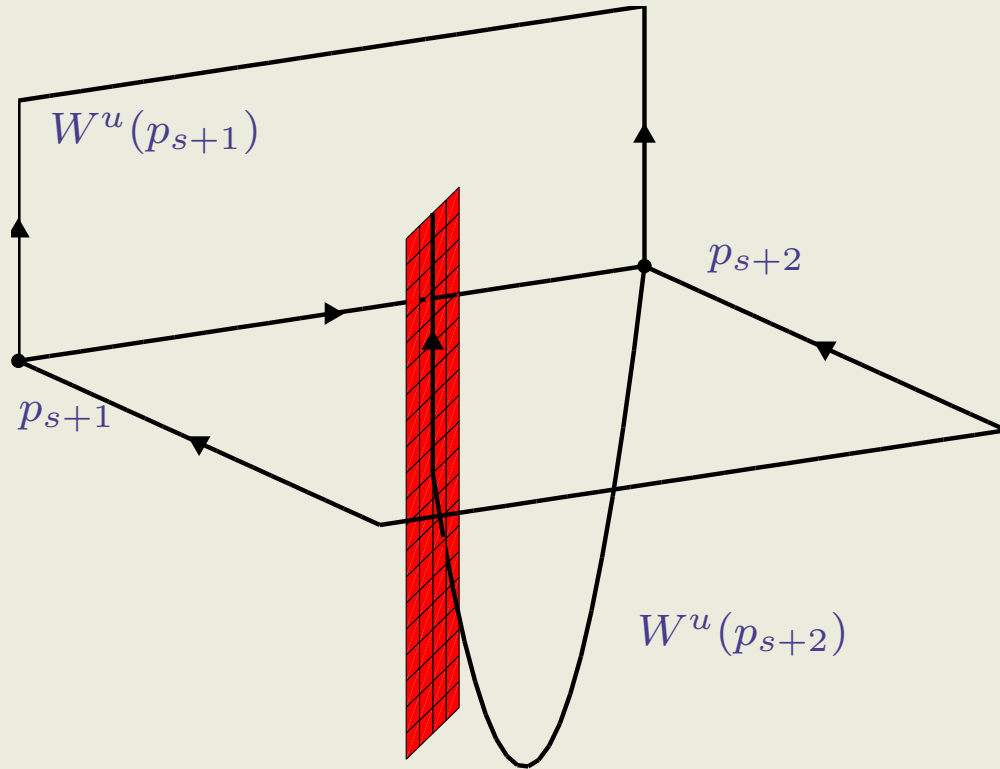


$$C^+ = x : \lambda^+(x) > 0$$

$$C^- = x : \lambda^-(x) < 0$$

$$m(C^+ \cup C^-) = 1$$

ergodicity



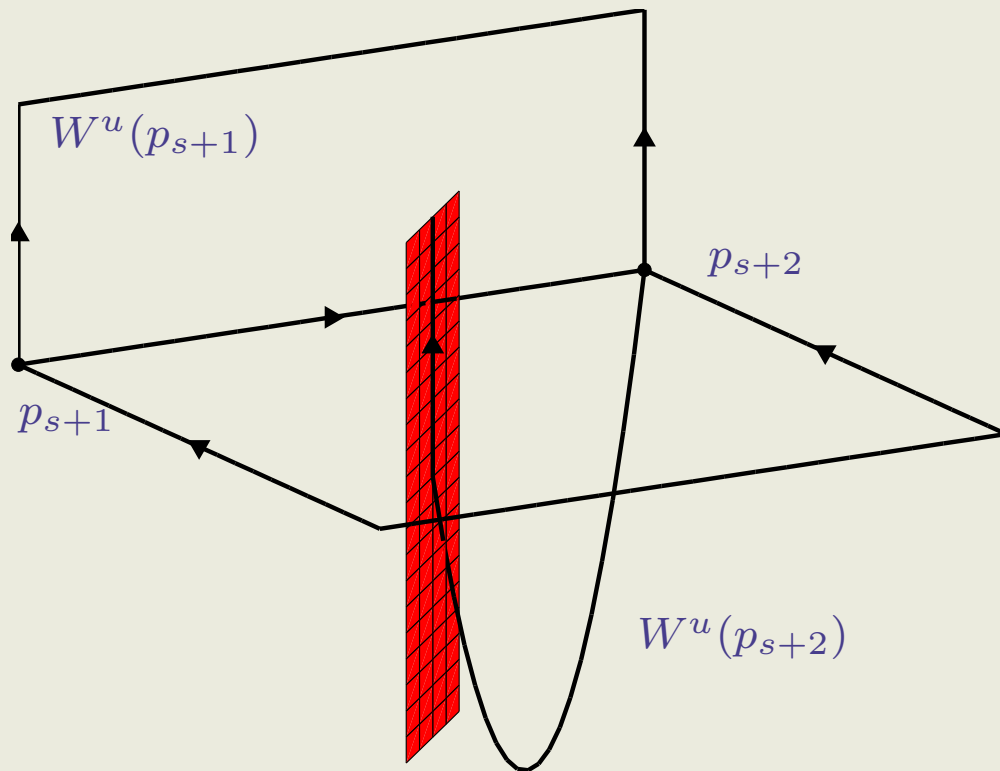
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$$C^+ \subset \Lambda^u(p_{s+1})$$

ergodicity



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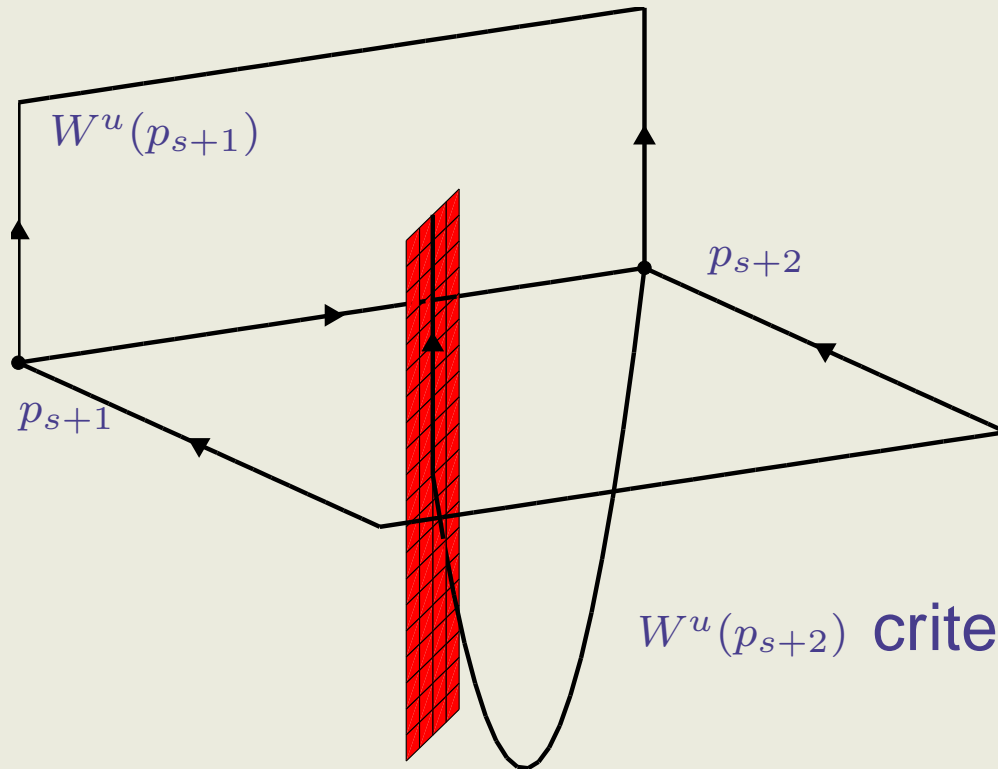
$$C^- = x : \lambda^-(x) < 0$$

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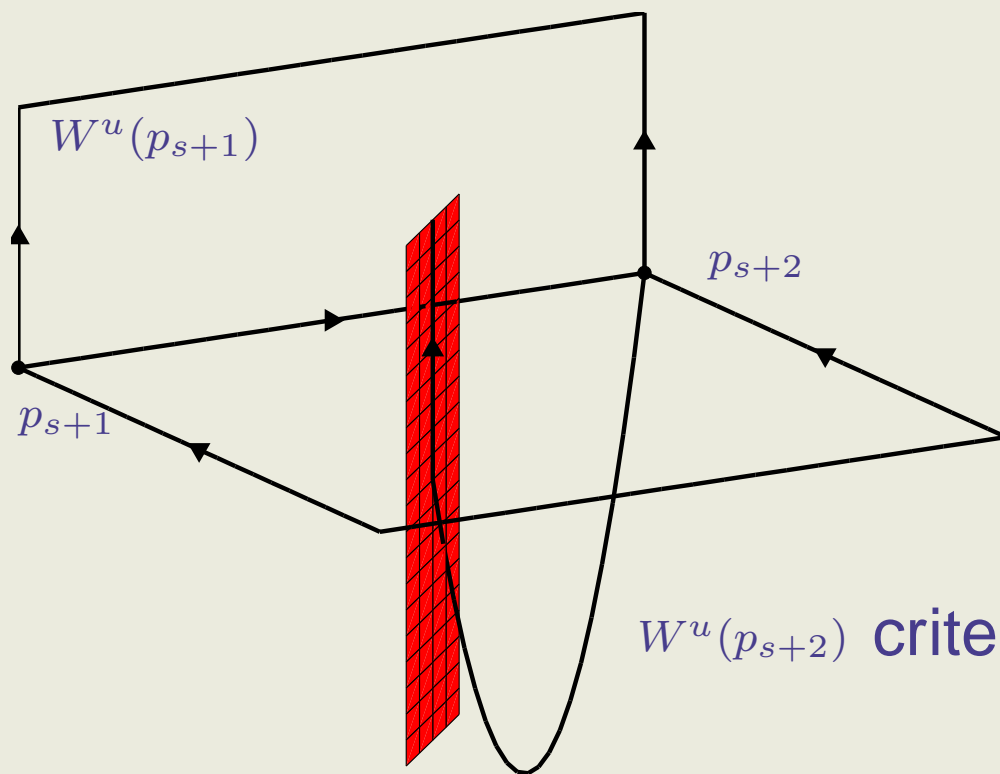
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$$C^+ \subset \Lambda^u(p_{s+1})$$

$$C^- \subset \Lambda^s(p_{s+1})$$

$W^u(p_{s+2})$ criterion applies

ergodicity



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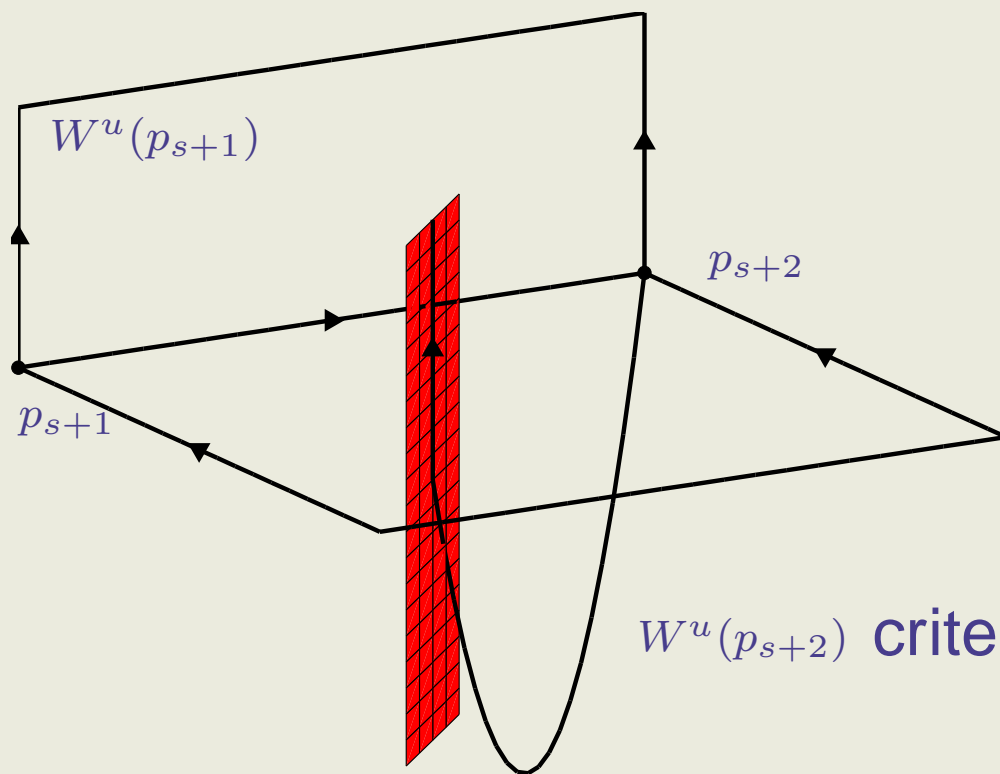
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$\Rightarrow g$ is ergodic

ergodicity



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$\Rightarrow g$ is ergodic \square