Partially hyperbolic dynamics

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program

partial hyperbolicity

- stable ergodicity

program

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- stable ergodicity
- partial hyperbolicity in 3D

program

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- stable ergodicity
- partial hyperbolicity in 3D
- ergodicity criteria

program

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- dynamical coherence

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- entropy

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setting

- M closed Riemannian manifold

setting

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- $f : M \to M$ diffeomorphism

setting

- M closed Riemannian manifold
- $f : M \to M$ diffeomorphism
- f preserves m smooth measure

ergodicity

ergodicity

- $f : M \to M$ is ergodic if

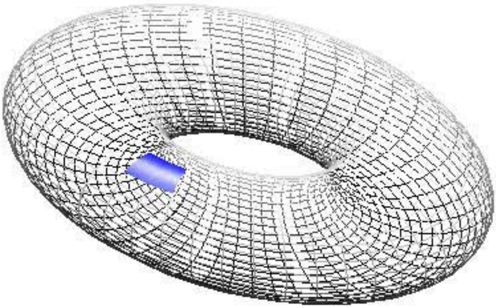
ergodicity

ergodicity

- $f : M \to M$ is ergodic if
- the orbit of every positive measure set “fills" M

ergodicity

an ergodic system



formulations of ergodicity

1

$$m(V) > 0$$

formulations of ergodicity

1

$$m(V) > 0 \quad \Longrightarrow \quad m\left(\bigcup_{n\geq 0} f^n(V)\right) = 1$$

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2

$\phi \in L^1_m(M)$ f-invariant

formulations of ergodicity

1

$$m(V) > 0 \quad \Longrightarrow \quad m\Big(\bigcup_{n\geq 0} f^n(V)\Big) = 1$$

2

$$\phi \in L^1_m(M) \quad f\text{-invariant} \quad \Longrightarrow \quad \phi \equiv \text{constant} \qquad \text{a.e.}$$

formulations of ergodicity

1

$$m(V) > 0 \quad \Longrightarrow \quad m\Big(\bigcup_{n\geq 0} f^n(V)\Big) = 1$$

2

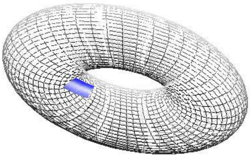
$$\phi \in L^1_m(M) \quad f\text{-invariant} \quad \Longrightarrow \quad \phi \equiv \text{constant} \qquad \text{a.e.}$$

3

$$\forall \phi \in L^1 \qquad \frac{1}{N}\sum_{n=0}^{N-1} \phi \circ f^n(x) \to \int \phi\, dm \qquad \text{a.e.}x$$

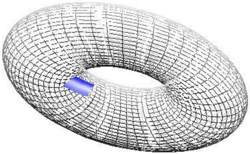
non-stable ergodicity

ergodic (irrational translation)

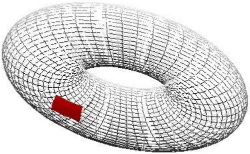


non-stable ergodicity

ergodic (irrational translation)



non-ergodic (rational translation)



stable ergodicity

stable ergodicity

- $f : M \to M$ is stably ergodic if

stable ergodicity

stable ergodicity

- $f : M \to M$ is stably ergodic if
- $$g \sim f \quad \Longrightarrow \quad g \quad \text{ergodic}$$

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- there is $\mathcal{U}_f \subset \mathrm{Diff}^1_m(M)$ such that

stable ergodicity

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- $f : M \to M$ is stably ergodic if
- there is $\mathcal{U}_f \subset \mathrm{Diff}^1_m(M)$ such that
-
$$g \in \mathcal{U} \cap \mathrm{Diff}^2_m(M) \quad \Longrightarrow \quad g \quad \text{ergodic}$$

problem

problem

(?) $\Longrightarrow$ stable ergodicity

hyperbolicity

hyperbolicity

$f : M \to M$ is hyperbolic if

$$TM = E^s \oplus E^u$$

hyperbolicity

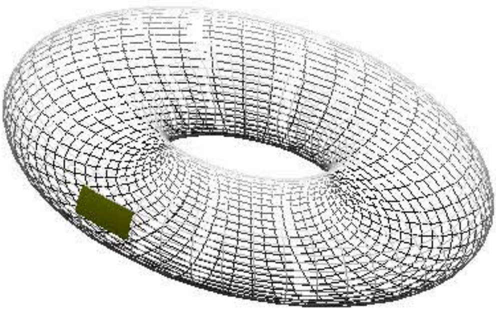
hyperbolicity

$f : M \to M$ is hyperbolic if

$$TM = \underset{\substack{\uparrow \\ \text{contracting}}}{E^s} \oplus \underset{\substack{\uparrow \\ \text{expanding}}}{E^u}$$

hyperbolicity $\Longrightarrow$ stable ergodicity

Anosov (1967)



problem

problem

$$(?) \quad \Longrightarrow \quad \text{stable ergodicity}$$

example

Grayson-Pugh-Shub (1994)

example

Grayson-Pugh-Shub (1994)

- stable ergodicity $\not\Longrightarrow$ hyperbolicity

example

Grayson-Pugh-Shub (1994)

- stable ergodicity $\not\Longrightarrow$ hyperbolicity
- partially hyperbolic

partial hyperbolicity

Definition

$f : M \to M$ is partially hyperbolic

$$TM = E^s \oplus E^c \oplus E^u$$

partial hyperbolicity

Definition

$f : M \to M$ is partially hyperbolic

$$\begin{array}{ccccccc} TM & = & E^s & \oplus & E^c & \oplus & E^u \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \text{contracting} & & \text{intermediate} & & \text{expanding} \end{array}$$

example

example

$$f : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1$$

example

example

$$f : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1$$

such that

$$f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times id$$

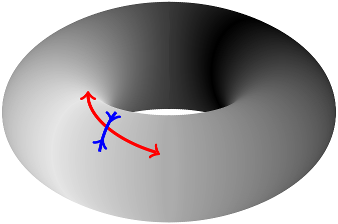
example

example

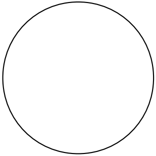
$$f : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1$$

such that

$$f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times id$$



$\times$



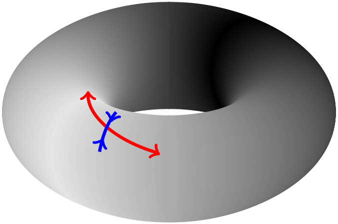
example

example

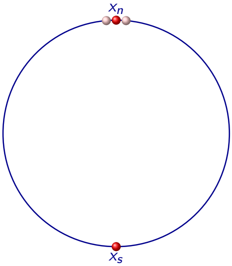
$$f : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1$$

such that

$$f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times NPSP$$



$\times$



remark

- in both examples, there is a foliation tangent to $E^s \oplus E^u$.

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- however, there are C^∞ perturbations such that

remark

- in both examples, there is a foliation tangent to $E^s \oplus E^u$.
- however, there are C^∞ perturbations such that
- there is no leaf tangent to $E^s \oplus E^u$ [HHU], [BHHTU]

conjecture

Pugh-Shub (1995)

partial hyperbolicity “$\Longrightarrow$" stable ergodicity

conjecture

Pugh-Shub (1995)

Stable ergodicity is C^r-dense among partially hyperbolic systems

result

Hertz-Hertz-Ures 08

Stable ergodicity is C^∞-dense among partially hyperbolic systems with $\dim E^c = 1$

result

Hertz-Hertz-Tahzibi-Ures 11

Stable ergodicity is C^1-dense among partially hyperbolic systems with $\dim E^c = 2$

program

Pugh-Shub program

program

Pugh-Shub program

program

Pugh-Shub program

conjecture A

Accessibility $\Longrightarrow$ ergodicity

program

Pugh-Shub program

conjecture A

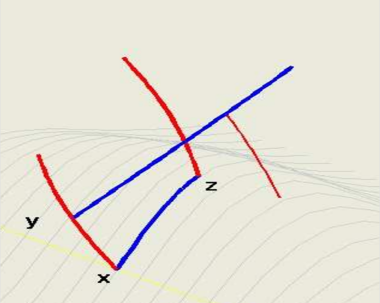
Accessibility $\Longrightarrow$ ergodicity

conjecture B

Stable accessibility is C^r-dense among partially hyperbolic systems

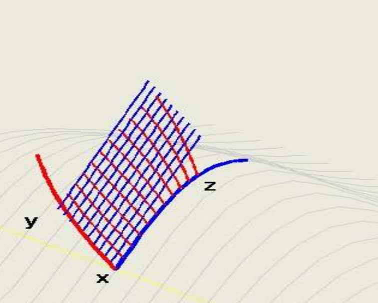
accessibility class

accessibility class



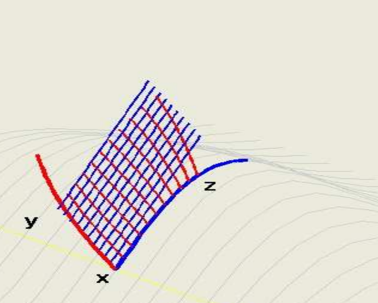
accessibility

example (non-accessible)

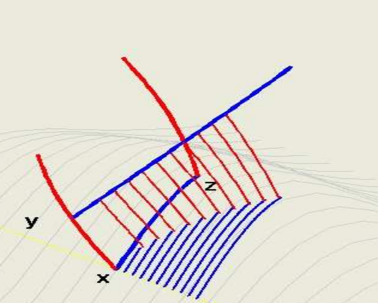


accessibility

example (non-accessible)



example (accessible)



conjecture A

Burns-Dolgopyat-Pesin 02

$f : M \to M$ partially hyperbolic

conjecture A

Burns-Dolgopyat-Pesin 02

$f : M \to M$ partially hyperbolic

- accessibility

conjecture A

Burns-Dolgopyat-Pesin 02

$f : M \to M$ partially hyperbolic

- accessibility
- center Lyapunov exponents > 0

conjecture A

Burns-Dolgopyat-Pesin 02

$f : M \to M$ partially hyperbolic

- accessibility
- center Lyapunov exponents > 0

$\Rightarrow$ stable ergodicity

conjecture A

Burns-Wilkinson 10

$f : M \to M$ partially hyperbolic

conjecture A

Burns-Wilkinson 10

$f : M \to M$ partially hyperbolic

- accessibility

conjecture A

Burns-Wilkinson 10

$f : M \to M$ partially hyperbolic

- accessibility
- bunching condition

conjecture A

Burns-Wilkinson 10

$f : M \to M$ partially hyperbolic

- accessibility
- bunching condition

$\Rightarrow$ ergodicity

conjecture B

Dolgopyat-Wilkinson 03

Stable accessibility is C^1-dense among $\mathcal{P}H_*(M)$

conjecture B

Didier 03

Accessibility is C^1-open if $\dim E^c = 1$

conjecture B

Hertz-Hertz-Ures 08, Burns-HH-Talitskaya-U 08

- $\dim E^c = 1$

conjecture B

Hertz-Hertz-Ures 08, Burns-HH-Talitskaya-U 08

- dim $E^c = 1$

Stable accessibility is C^∞ dense in $\mathcal{P}H_*(M)$

remark

Tahzibi

stable ergodicity $\not\Rightarrow$ partial hyperbolicity

remark

Arbieto-Matheus

stable ergodicity $\Rightarrow$ dominated splitting

domination

dominated splitting

$f : M \to M$ has a dominated splitting if

domination

dominated splitting

$f : M \to M$ has a dominated splitting if

$$TM = E \oplus F$$

domination

dominated splitting

$f : M \to M$ has a dominated splitting if

$$TM = \underset{\substack{\uparrow \\ Tf \text{ more contracting}}}{E} \oplus \underset{\substack{\uparrow \\ Tf \text{ more expanding}}}{F}$$

question

question

is accessibility condition open when $c > 1$?

question

question

are accessibility classes manifolds? (HHU 08 $c = 1$)

question

question

is stable accessibility dense in $\mathcal{P}H^{\omega}_{*}(M)$?

question

question

stable ergodicity C^1-dense in $\mathcal{P}H_m(M)$ with $c > 2$?

question

question

domination + (?) $\Rightarrow$ ergodicity?