Partially hyperbolic dynamics

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program

partial hyperbolicity

- stable ergodicity

program

partial hyperbolicity

- stable ergodicity
- partial hyperbolicity in 3D

program

partial hyperbolicity

- stable ergodicity
- partial hyperbolicity in 3D
- ergodicity criteria

program

partial hyperbolicity

- stable ergodicity
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- ergodicity criteria
- dynamical coherence

program

partial hyperbolicity

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- entropy

program

partial hyperbolicity

- stable ergodicity
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- entropy

setting

- M^3 closed 3-manifold

setting

- M^3 closed 3-manifold
- $f : M \to M$ diffeomorphism

partial hyperbolicity

partial hyperbolicity

$f : M \to M$ is partially hyperbolic

$$TM = E^s \oplus E^c \oplus E^u$$

partial hyperbolicity

partial hyperbolicity

$f : M \to M$ is partially hyperbolic

$$TM = E^s \oplus E^c \oplus E^u$$

$E^s \uparrow$ contracting, $E^c \uparrow$ intermediate, $E^u \uparrow$ expanding

partial hyperbolicity

partial hyperbolicity

$f : M \to M$ is partially hyperbolic

$$\begin{array}{ccccccc} TM & = & E^s & \oplus & E^c & \oplus & E^u \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 1-\mathrm{dim} & & 1-\mathrm{dim} & & 1-\mathrm{dim} \end{array}$$

example

example

$$f : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1$$

example

example

$$f : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1$$

such that

$$f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times id$$

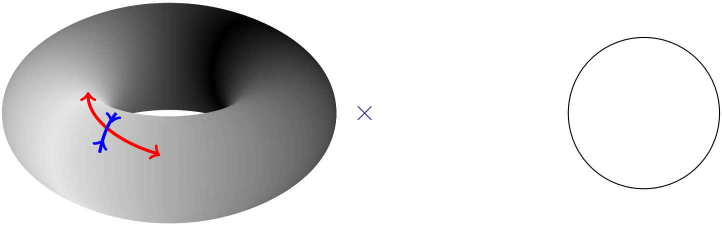
example

example

$$f : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1$$

such that

$$f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times id$$



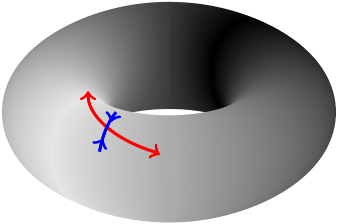
example

example

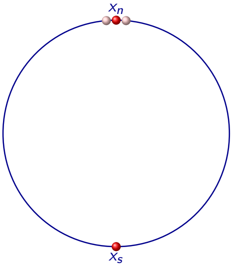
$$f : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1$$

such that

$$f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times NPSP$$



$\times$



problems

- ergodicity

problems

- ergodicity
- dynamical coherence

problems

- ergodicity
- dynamical coherence
- classification

problems

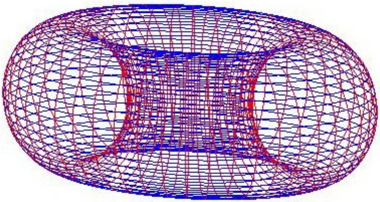
- ergodicity
- dynamical coherence
- classification

$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \rightarrow$ Anosov torus

Anosov torus

Anosov torus

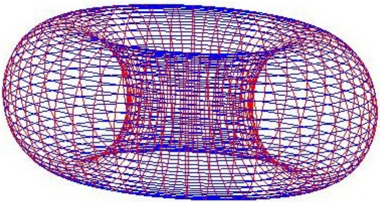
- T embedded 2-torus



Anosov torus

Anosov torus

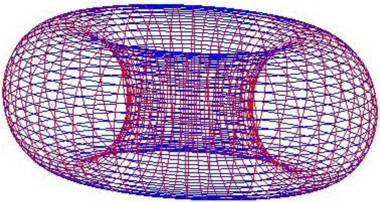
- T embedded 2-torus
- $\exists f : M \to M$ s.t.



Anosov torus

Anosov torus

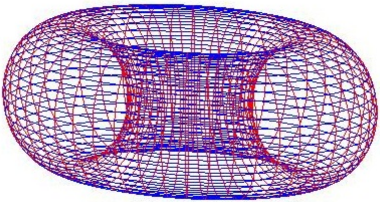
- T embedded 2-torus
- $\exists f : M \to M$ s.t.
  1. $f(T) = T$



Anosov torus

Anosov torus

- T embedded 2-torus
- $\exists f : M \to M$ s.t.
  1. $f(T) = T$
  2. $f|_T$ isotopic to Anosov



remark

remark

$f : M \to M$ not necessarily partially hyperbolic!

remark

lemma

- T Anosov torus

remark

lemma

- T Anosov torus
- $\Longrightarrow \exists\, F : M \to M$

remark

lemma

- T Anosov torus
- $\Longrightarrow \exists\, F : M \to M$
- $A = F|_T$ hyperbolic automorphism

example

example

- $f : M \to M$ partially hyperbolic

example

example

- $f : M \to M$ partially hyperbolic
- p periodic point

example

example

- $f : M \to M$ partially hyperbolic
- p periodic point
- $AC(p)$ compact

example

example

- $f : M \to M$ partially hyperbolic
- p periodic point
- $AC(p)$ compact
- $\implies AC(p)$ is an Anosov torus

remark

remark

- $f : M \to M$ partially hyperbolic

remark

remark

- $f : M \to M$ partially hyperbolic
- $AC(x)$ compact for some $x \in M$

remark

remark

- $f : M \to M$ partially hyperbolic
- $AC(x)$ compact for some $x \in M$
- $\implies \exists$ Anosov torus in M

theorem

theorem (HHU11)

If an irreducible M contains an Anosov torus,

theorem

theorem (HHU11)

If an irreducible M contains an Anosov torus, then M is either

theorem

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If an irreducible M contains an Anosov torus, then M is either

1. $\mathbb{T}^3$

theorem

theorem (HHU11)

If an irreducible M contains an Anosov torus, then M is either

1. $\mathbb{T}^3$
2. the mapping torus of $-id : \mathbb{T}^2 \to \mathbb{T}^2$

theorem

theorem (HHU11)

If an irreducible *M* contains an Anosov torus, then *M* is either

1. $\mathbb{T}^3$
2. the mapping torus of $-id : \mathbb{T}^2 \to \mathbb{T}^2$
3. a mapping torus of a hyperbolic automorphism of $\mathbb{T}^2$

remark

remark

- $f : M \to M$ partially hyperbolic

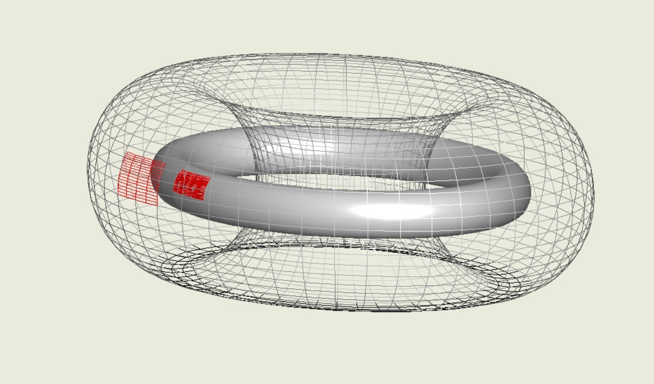
remark

remark

- $f : M \to M$ partially hyperbolic
- $\Longrightarrow$ M is irreducible (Burago-Ivanov08, Rosenberg68)

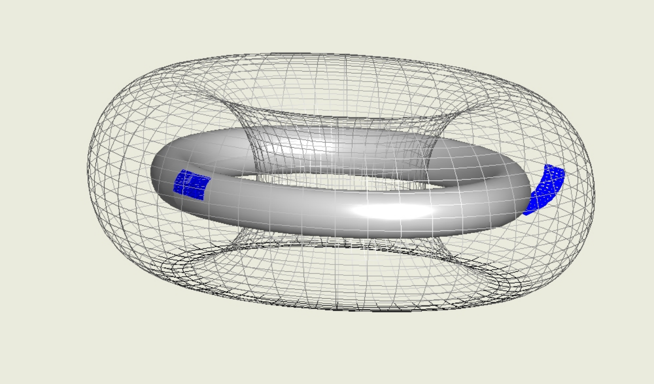
manifolds with Anosov tori

1 $\mathbb{T}^3$



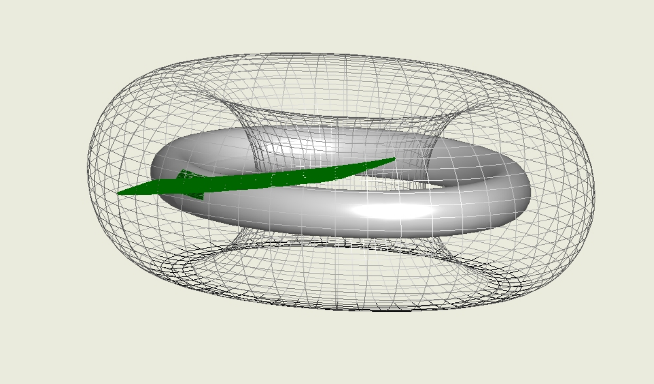
manifolds with Anosov tori

2 mapping torus of $-id$



manifolds with Anosov tori

3 mapping torus of hyperbolic automorphism



remark

remark

- $f : M \to M$ partially hyperbolic

remark

remark

- $f : M \to M$ partially hyperbolic
- $\Longrightarrow$ only these manifolds contain compact accessibility classes

almost all ph are ergodic

Hertz-Hertz-Ures 08

Stable ergodicity is C^{∞}-dense among partially hyperbolic systems of M^3

problem

problem

describe non-ergodic partially hyperbolic systems

question

question

$\exists$ M^3 such that all p.h. in M are ergodic?

question

question

$\exists$ M^3 such that all p.h. in M are ergodic?

YES

result

Hertz-Hertz-Ures 08

- N^3 nilmanifold different from $\mathbb{T}^3$

result

Hertz-Hertz-Ures 08

- N^3 nilmanifold different from $\mathbb{T}^3$
- $\Longrightarrow$ all p.h. in N^3 are ergodic

question

question

general case?

non-ergodic conjecture

non-ergodic conjecture (HHU)

- $f : M^3 \to M^3$ non-ergodic conservative partially hyperbolic,

non-ergodic conjecture

non-ergodic conjecture (HHU)

- $f : M^3 \to M^3$ non-ergodic conservative partially hyperbolic,
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harder question

f non-ergodic $\overset{?}{\Longrightarrow}$ compact accessibility class

integrability

integrability

$f : M \to M$ partially hyperbolic

$$TM = E^s \oplus E^c \oplus E^u$$

integrability

integrability

$f : M \to M$ partially hyperbolic

$$\begin{array}{ccccccc} TM & = & E^s & \oplus & E^c & \oplus & E^u \\ & & \uparrow & & & & \uparrow \\ & & \mathcal{F}^s & & & & \mathcal{F}^u \end{array}$$

integrability

integrability

$f : M \to M$ partially hyperbolic

$$\begin{array}{ccccccc} TM & = & E^s & \oplus & E^c & \oplus & E^u \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \mathcal{F}^s & & \text{?} & & \mathcal{F}^u \end{array}$$

remark

remark

- $\mathcal{F}^s$ and $\mathcal{F}^u$ are unique

remark

remark

- $\mathcal{F}^s$ and $\mathcal{F}^u$ are unique
- $\mathcal{F}^s$ and $\mathcal{F}^u$ are f-invariant

dynamical coherence

dynamical coherence

dynamical coherence

dynamical coherence

- $f : M \to M$ partially hyperbolic

dynamical coherence

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- $f : M \to M$ partially hyperbolic
- is dynamically coherent if

dynamical coherence

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- $f : M \to M$ partially hyperbolic
- is dynamically coherent if
  1. $\exists$ invariant foliation $\mathcal{F}^{cs}$ tangent to $E^s \oplus E^c$

dynamical coherence

dynamical coherence

- $f : M \to M$ partially hyperbolic
- is dynamically coherent if
  1. $\exists$ invariant foliation $\mathcal{F}^{cs}$ tangent to $E^s \oplus E^c$
  2. $\exists$ invariant foliation $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$

remark

remark

- If f is dynamically coherent, then

remark

remark

- If f is dynamically coherent, then
- $\exists$ invariant foliation $\mathcal{F}^c$

remark

remark

- If f is dynamically coherent, then
- $\exists$ invariant foliation $\mathcal{F}^c$
- that sub-foliates $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$

remark

remark

- If f is dynamically coherent, then
- $\exists$ invariant foliation $\mathcal{F}^c$
- that sub-foliates $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$

remark

$\mathcal{F}^c$ not necessarily unique!

remark

remark

- $W^{uc}(p)$ is a compact leaf tangent to $E^c \oplus E^u$

remark

remark

- $W^{uc}(p)$ is a compact leaf tangent to $E^c \oplus E^u$
- p is a periodic point

remark

remark

- $W^{uc}(p)$ is a compact leaf tangent to $E^c \oplus E^u$
- p is a periodic point
- $\implies W^{uc}(p)$ is an Anosov torus

remark

remark

- $W^{uc}(x)$ is a compact leaf for any $x \in M$

remark

remark

- $W^{uc}(x)$ is a compact leaf for any $x \in M$
- $\implies \exists$ Anosov torus in M

theorem

Hertz-Hertz-Ures 11

- $f : M \to M$ partially hyperbolic

theorem

Hertz-Hertz-Ures 11

- $f : M \to M$ partially hyperbolic
- $\exists$ invariant $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$

theorem

Hertz-Hertz-Ures 11

- $f : M \to M$ partially hyperbolic
- $\exists$ invariant $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$
- $\implies \mathcal{F}^{cu}$ does not contain compact leaves

non-dynamically coherent conjecture

non-dynamically coherent conjecture (HHU)

- $f : M^3 \to M^3$ non-dynamically coherent partially hyperbolic,

non-dynamically coherent conjecture

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harder question

f non-dynamically coherent $\stackrel{?}{\Longrightarrow}$

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non-dynamically coherent conjecture (HHU)

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harder question

f non-dynamically coherent $\overset{?}{\Longrightarrow}$

- exists a compact leaf tangent to $E^s \oplus E^c$ (repelling), or

non-dynamically coherent conjecture

non-dynamically coherent conjecture (HHU)

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  2. the mapping torus of $-id : \mathbb{T}^2 \to \mathbb{T}^2$
  3. the mapping torus of a hyperbolic map of $\mathbb{T}^2$

harder question

f non-dynamically coherent $\overset{?}{\Longrightarrow}$

- exists a compact leaf tangent to $E^s \oplus E^c$ (repelling), or
- exists a compact leaf tangent to $E^c \oplus E^u$ (attracting).

conjectures

conjectures

non-ergodic conjecture

$f : M \to M$ non-ergodic partially hyperbolic

conjectures

non-ergodic conjecture

$f : M \to M$ non-ergodic partially hyperbolic

non-dyn. coh. conjecture

$f : M \to M$ non-dyn. coherent partially hyperbolic

conjectures

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then M is either

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conjectures

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conjectures

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?

$\exists$ Anosov torus tangent to $E^s \oplus E^u$?

conjectures

non-ergodic conjecture

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non-dyn. coh. conjecture

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then M is either

1. $\mathbb{T}^3$
2. the mapping torus of $-id : \mathbb{T}^2 \to \mathbb{T}^2$
3. the mapping torus of a hyperbolic map of $\mathbb{T}^2$

?

$\exists$ Anosov torus tangent to $E^s \oplus E^u$?

?

$\exists$ Anosov torus tangent to $E^c \oplus E^u$ or $E^c \oplus E^c$?

examples of ph dynamics

known ph dynamics in dimension 3

examples of ph dynamics

known ph dynamics in dimension 3

- perturbations of time-one maps of Anosov flows

examples of ph dynamics

known ph dynamics in dimension 3

- perturbations of time-one maps of Anosov flows
- certain skew-products

examples of ph dynamics

known ph dynamics in dimension 3

- perturbations of time-one maps of Anosov flows
- certain skew-products
- certain DA-maps

question

question

are there more examples?

conjecture Pujals

conjecture (Pujals01)

If $f : M^3 \to M^3$ is a transitive partially hyperbolic diffeomorphism, then f is (finitely covered by) either

conjecture Pujals

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If $f : M^3 \to M^3$ is a transitive partially hyperbolic diffeomorphism, then f is (finitely covered by) either

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2. a skew-product
3. a DA-map

theorem

Bonatti-Wilkinson04

$f : M^3 \to M^3$ transitive and partially hyperbolic, such that

theorem

Bonatti-Wilkinson04

$f : M^3 \to M^3$ transitive and partially hyperbolic, such that

- $\exists \mathcal{C}$ invariant circle

theorem

Bonatti-Wilkinson04

$f : M^3 \to M^3$ transitive and partially hyperbolic, such that

- $\exists \mathcal{C}$ invariant circle
- $\exists R > 0$ such that $W^s_R(\mathcal{C}) \cap W^u_R(\mathcal{C}) \setminus \mathcal{C}$ contains a a circle

theorem

Bonatti-Wilkinson04

$f : M^3 \to M^3$ transitive and partially hyperbolic, such that

- $\exists \mathcal{C}$ invariant circle
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then

theorem

Bonatti-Wilkinson04

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- $\exists \mathcal{C}$ invariant circle
- $\exists R > 0$ such that $W^s_R(\mathcal{C}) \cap W^u_R(\mathcal{C}) \setminus \mathcal{C}$ contains a a circle

then

1. f is dynamically coherent

theorem

Bonatti-Wilkinson04

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- $\exists R > 0$ such that $W^s_R(\mathcal{C}) \cap W^u_R(\mathcal{C}) \setminus \mathcal{C}$ contains a a circle

then

1. f is dynamically coherent
2. f is finitely covered by a skew-product

conjecture

conjeture (HHU)

If $f : M^3 \to M^3$ is partially hyperbolic and dynamically coherent, then either

conjecture

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If $f : M^3 \to M^3$ is partially hyperbolic and dynamically coherent, then either

1. f^n is a perturbation of a time-one map of an Anosov flow,

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If $f : M^3 \to M^3$ is partially hyperbolic and dynamically coherent, then either

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conjecture

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Moreover, in case:

1. f^n is leafwise conjugate to an Anosov flow

conjecture

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Moreover, in case:

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2. f is leafwise conjugate to a skew-product with linear base

conjecture

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Moreover, in case:

1. f^n is leafwise conjugate to an Anosov flow
2. f is leafwise conjugate to a skew-product with linear base
3. f is leafwise conjugate to an Anosov map in $\mathbb{T}^3$.

theorem

Hammerlindl09

If $f : \mathbb{T}^3 \to \mathbb{T}^3$ is absolutely partially hyperbolic,

theorem

Hammerlindl09

If $f : \mathbb{T}^3 \to \mathbb{T}^3$ is absolutely partially hyperbolic,
then f is leafwise conjugate to a ph linear map

theorem

Hammerlind10

If $f : N^3 \to N^3$ is absolutely partially hyperbolic, N nilmanifold,

theorem

Hammerlind10

If $f : N^3 \to N^3$ is absolutely partially hyperbolic, N nilmanifold, then f is conjugate to a skew-product