New criteria for ergodicity and non-uniform hyperbolicity

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R. Ures

Smale’s Spectral Decomposition Theorem

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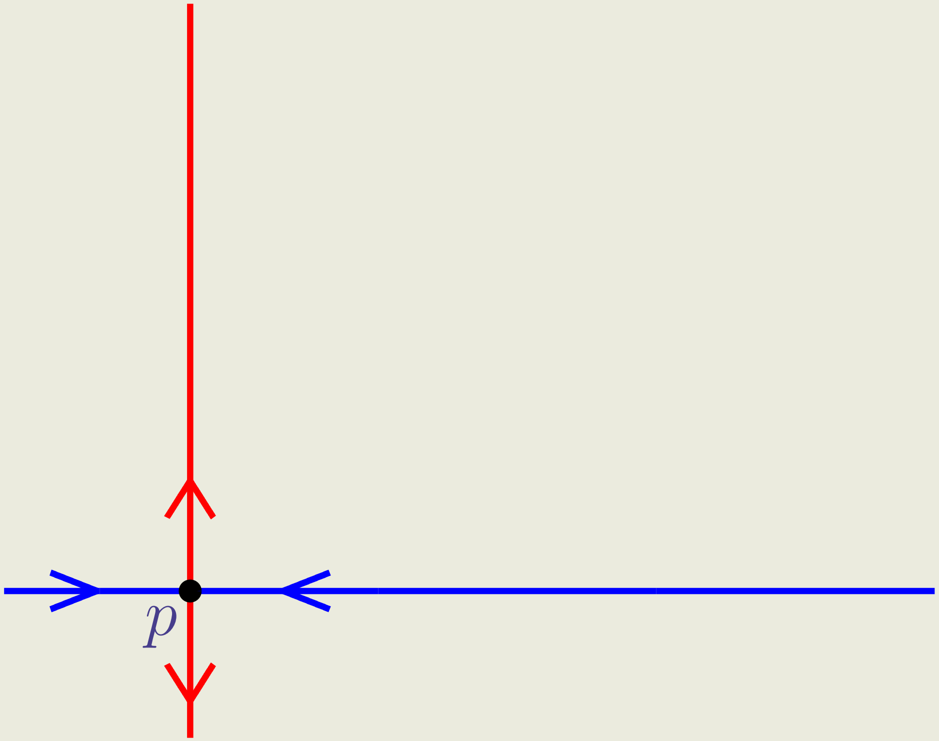
Moreover

$$\Lambda_i = \overline{H(p)}$$

Homoclinic class

$$q \in H(p) \qquad p \in \mathrm{Per}(f)$$

$$\Updownarrow$$



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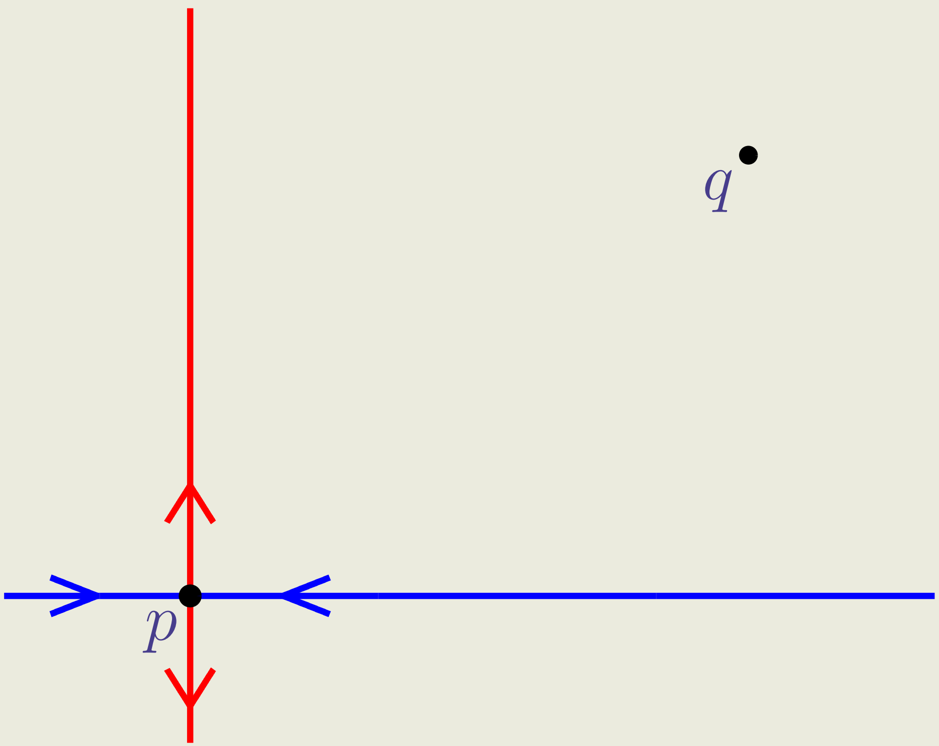

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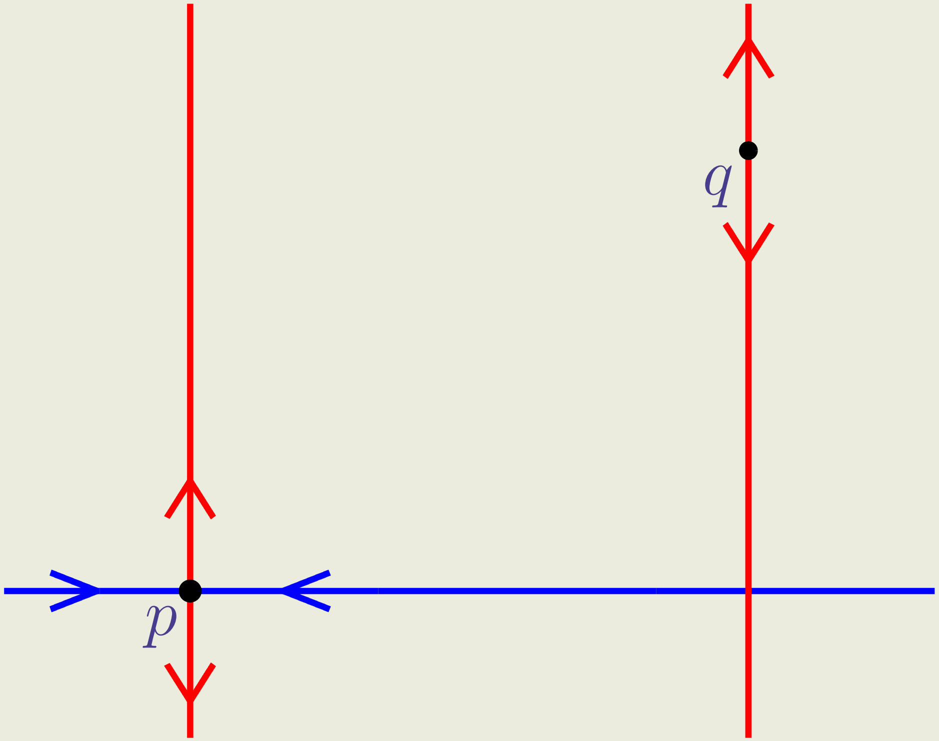

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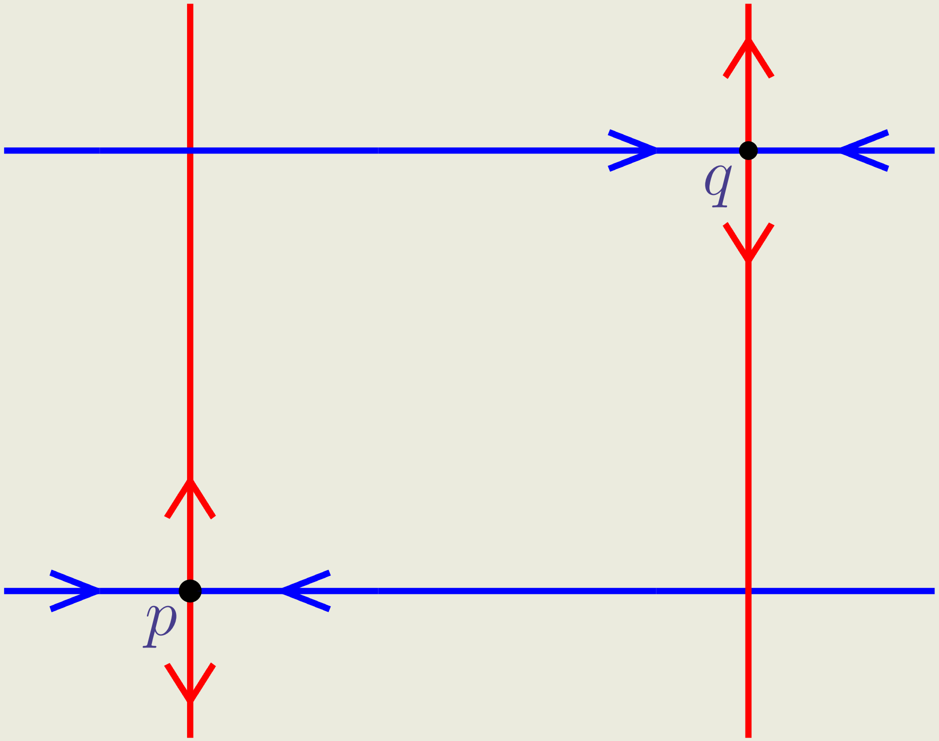

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Moreover

$$\Lambda_i \doteq B(p)$$

Pesin's invariant manifolds

$f \in \mathrm{Diff}^{1+}_m(M)$

Pesin's stable manifold of x

$$\widetilde{W}^s(x) = \left\{ y : \overline{\lim_{n\to\infty}} \frac{1}{n} \log d(f^n(x), f^n(y)) < 0 \right\}$$

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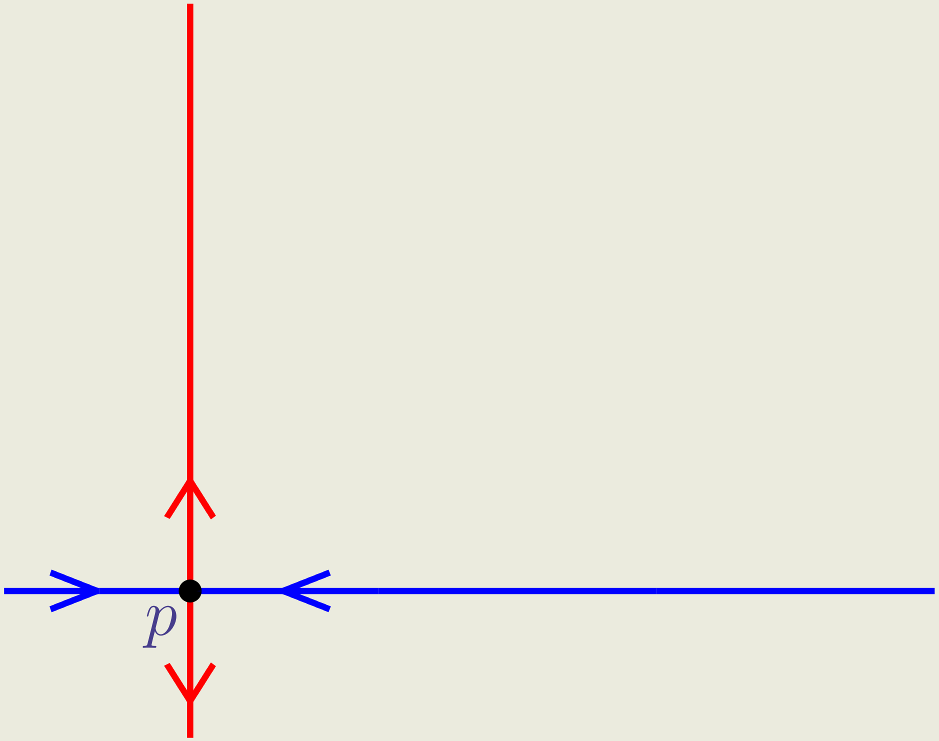

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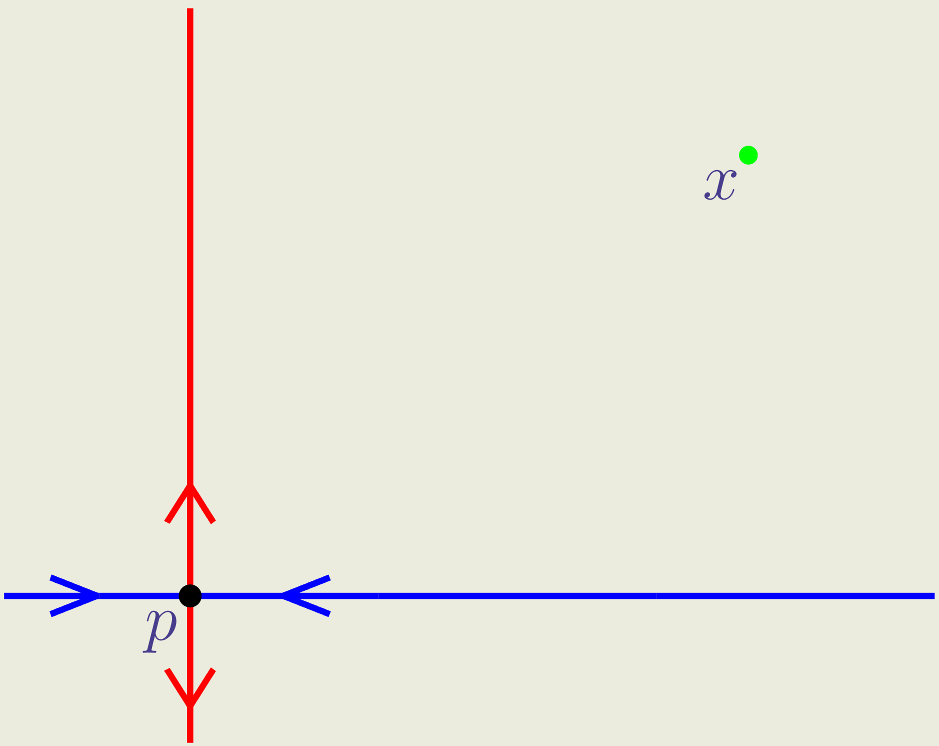

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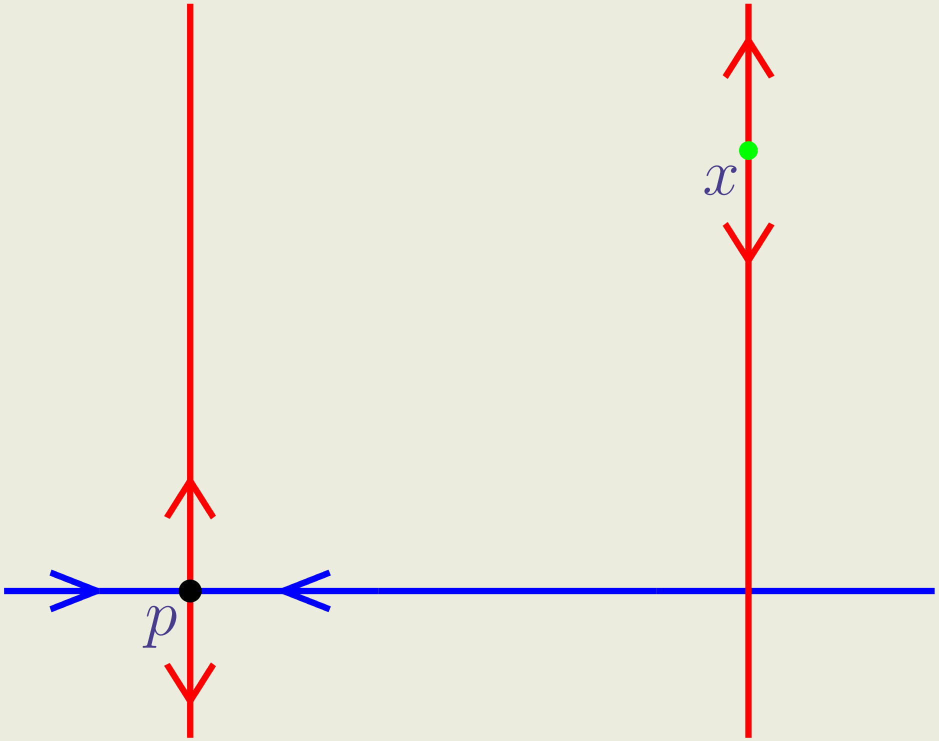

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$$\widetilde{W}^u(x) \pitchfork W^s(p) \neq \emptyset$$

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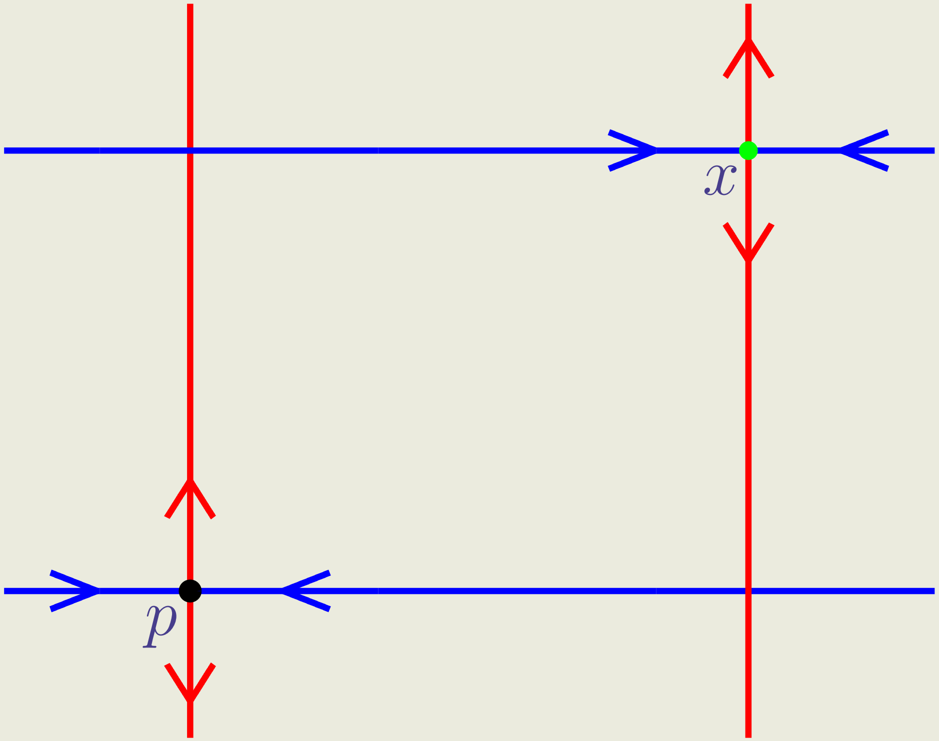

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$B^s(p)$ and $B^u(p)$

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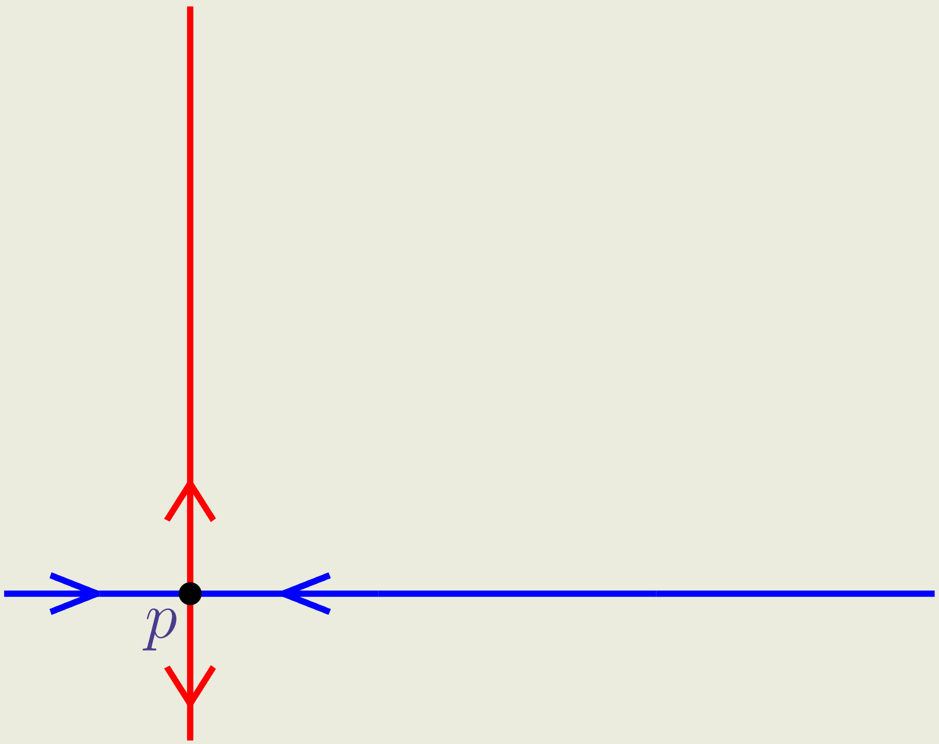

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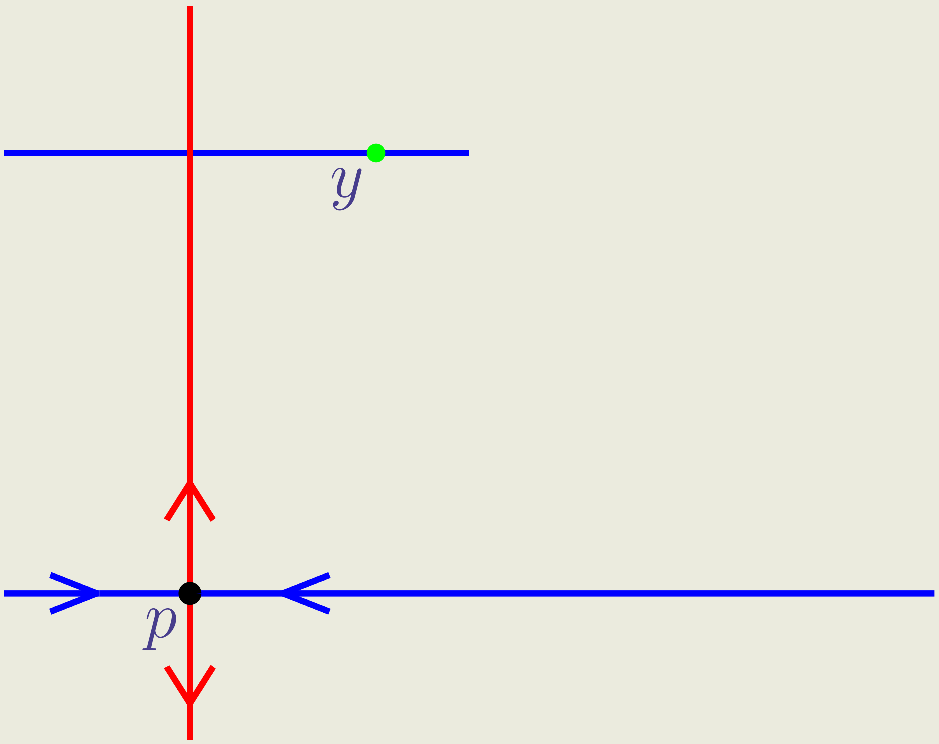

$$B^s(p) = \{y : \widetilde{W}^s(y) \pitchfork W^u(o(p))\}$$

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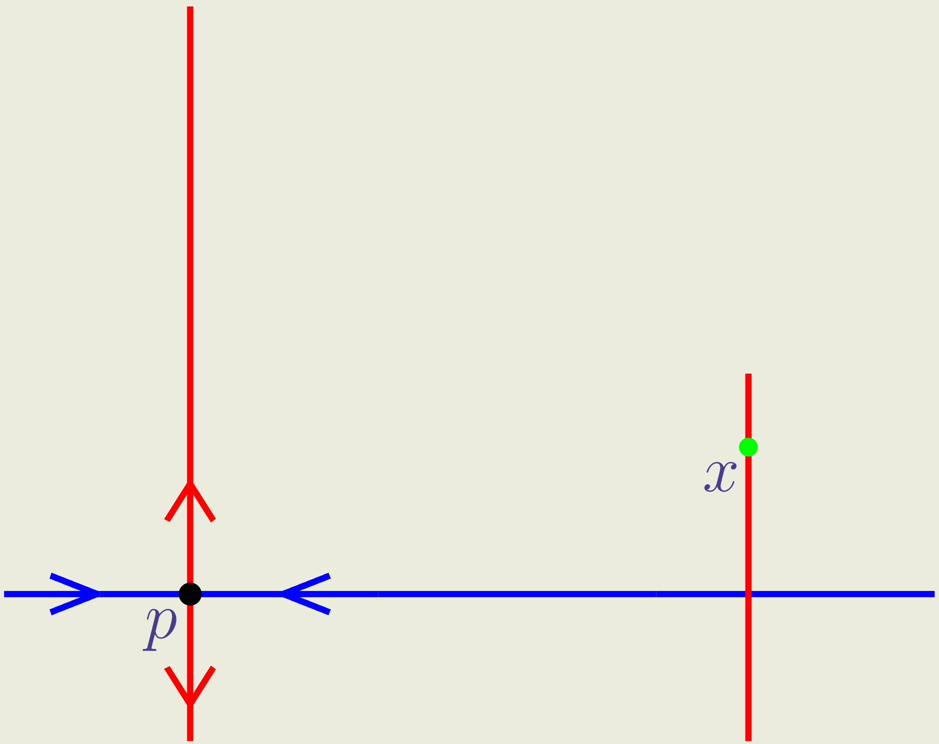

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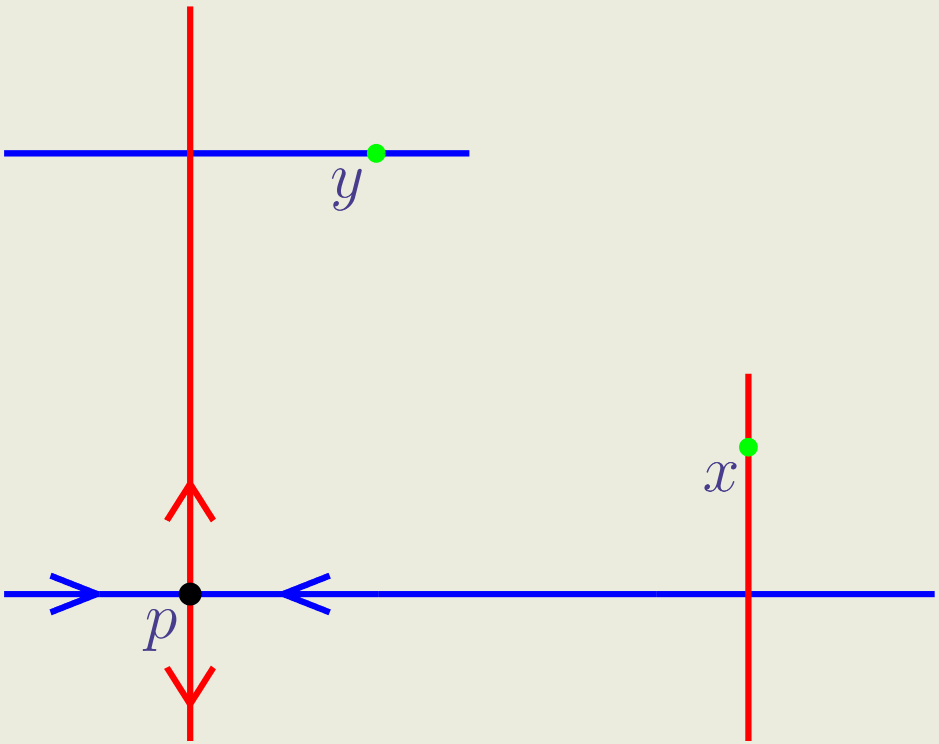

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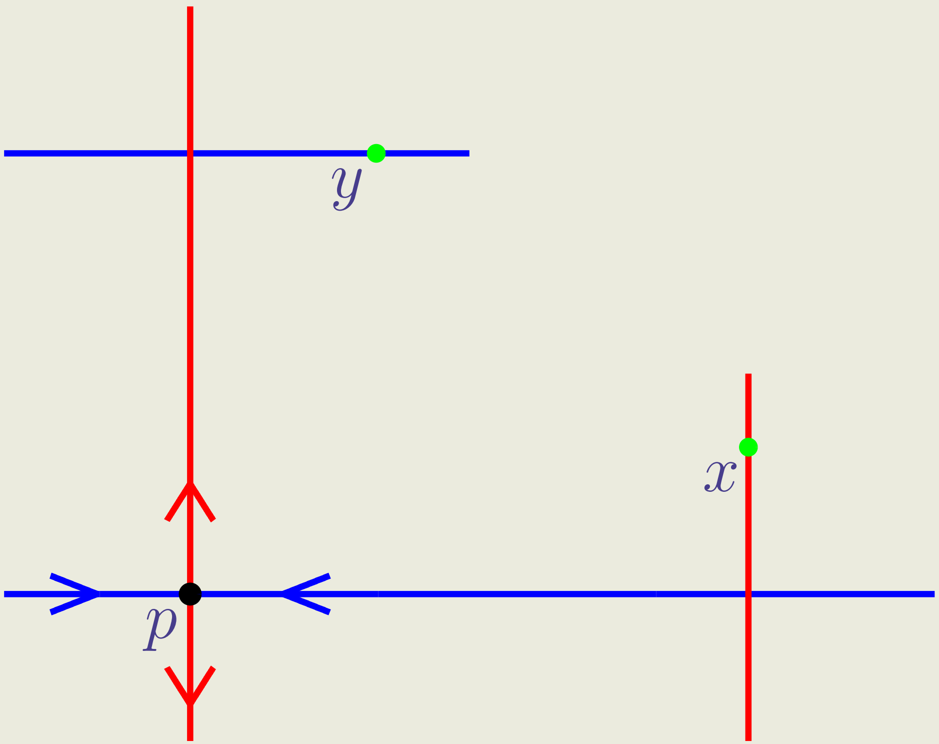

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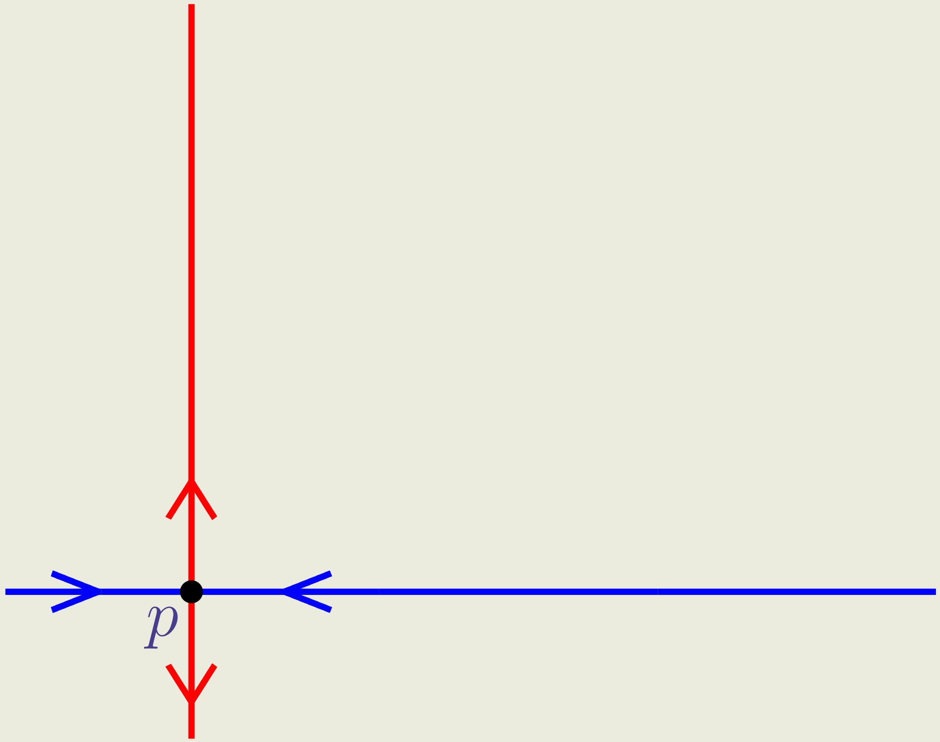
Ingredients

For each measurable set Λ there exists S_Λ full measure set s.t. for $x \in S_\Lambda$



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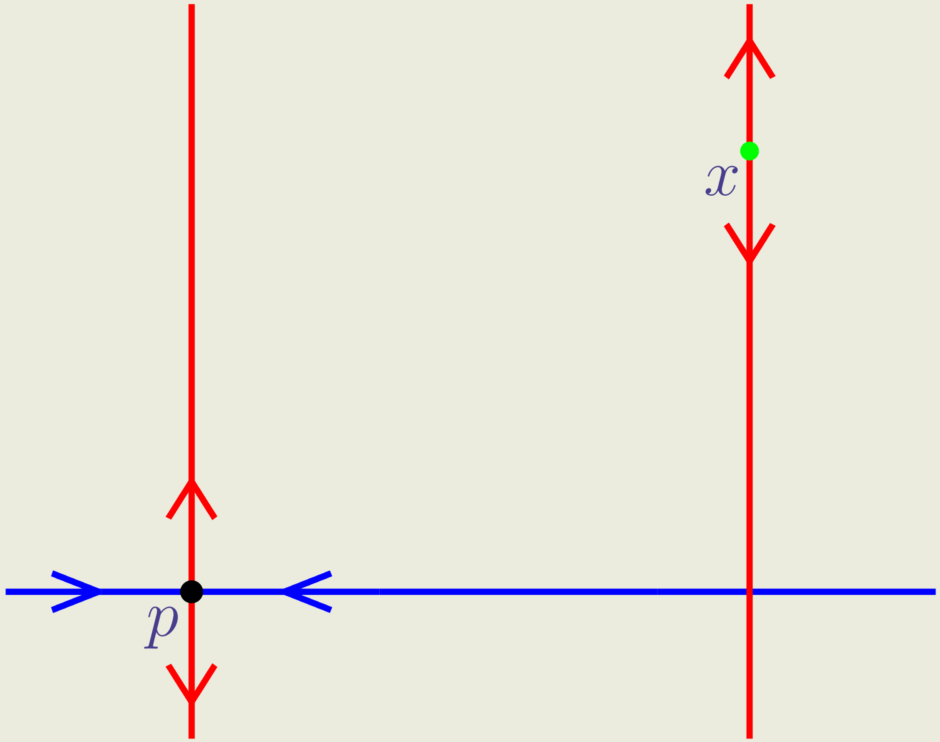

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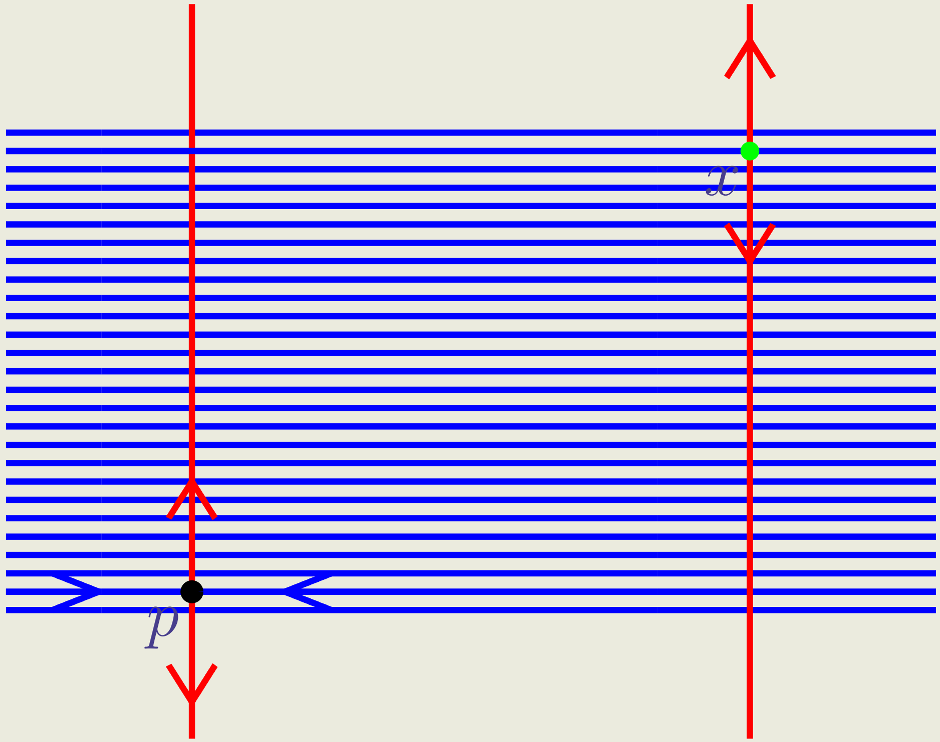

$$x \in \Lambda \Rightarrow \widetilde{W}^u(x) \mathring{\subset} \Lambda$$

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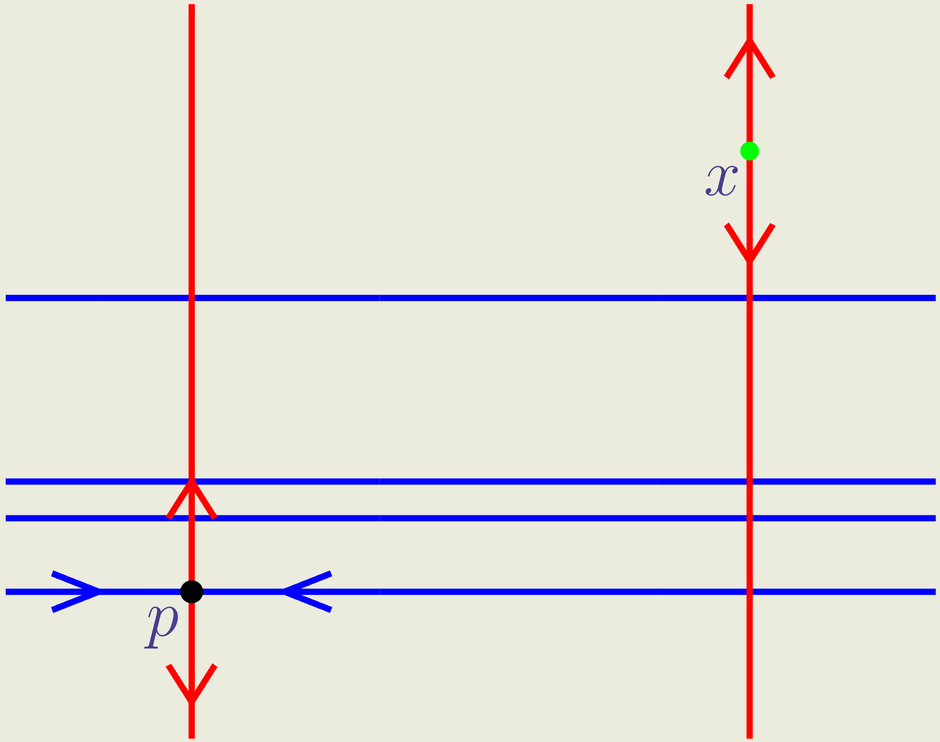
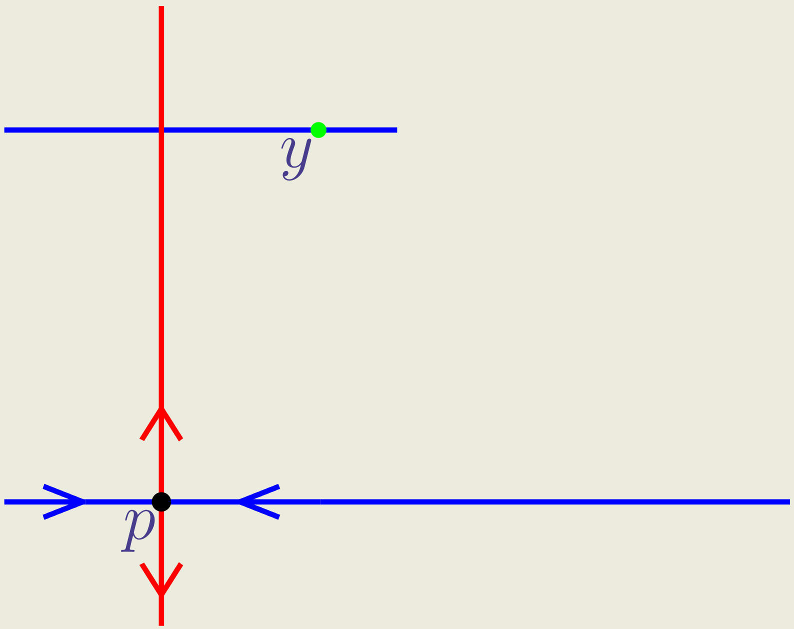


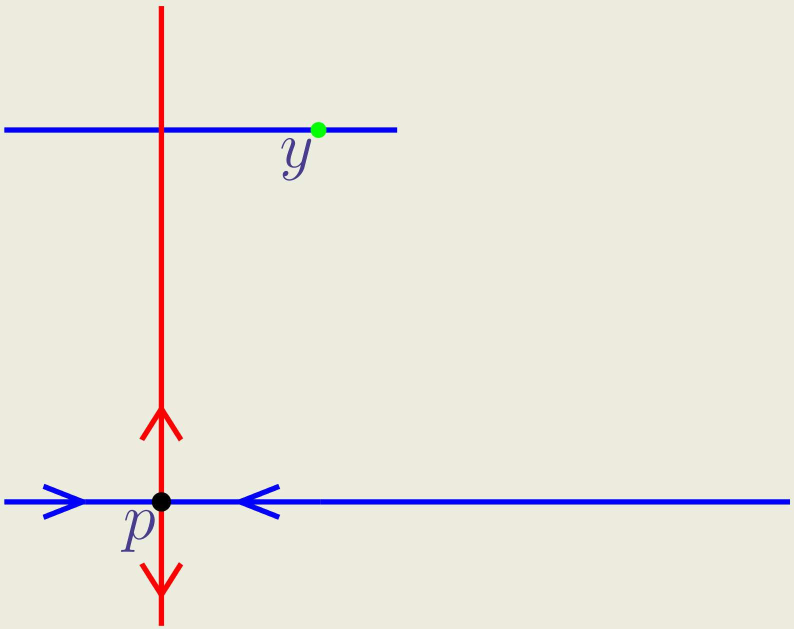
$$x \notin \Lambda \Rightarrow m_u(\widetilde{W^u}(x) \cap \Lambda) = 0$$

Ingredients



- $y \in B^s(p) \cap$ Pesin block

Ingredients



- $y \in B^s(p) \cap$ Pesin block
- $d(y, W^u(p)) < \varepsilon$

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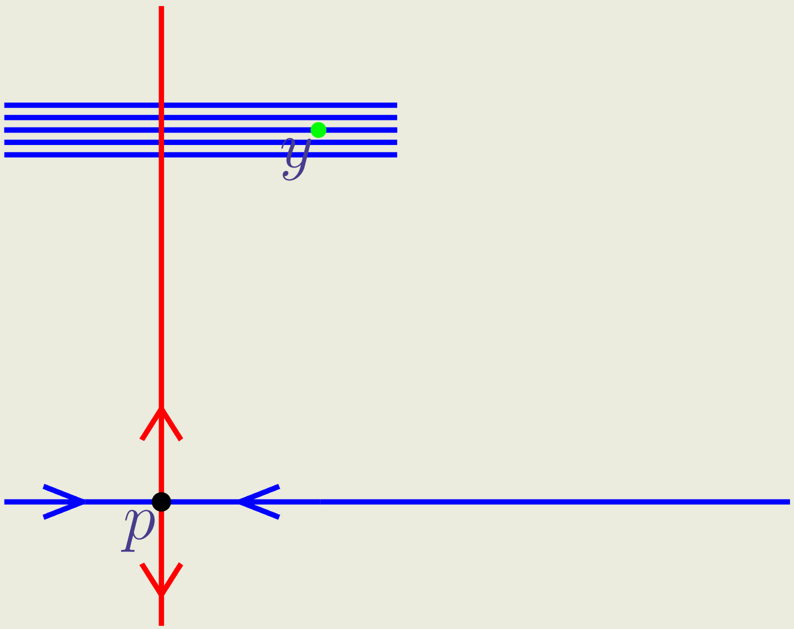

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- $\widetilde{W}^s_{loc}(w) \pitchfork W^u(p)$ for all w near y in the Pesin block

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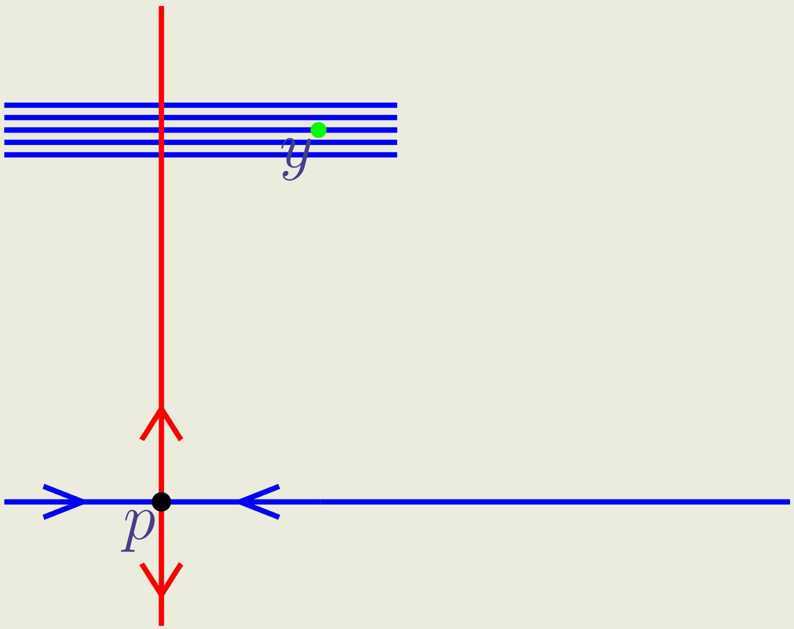

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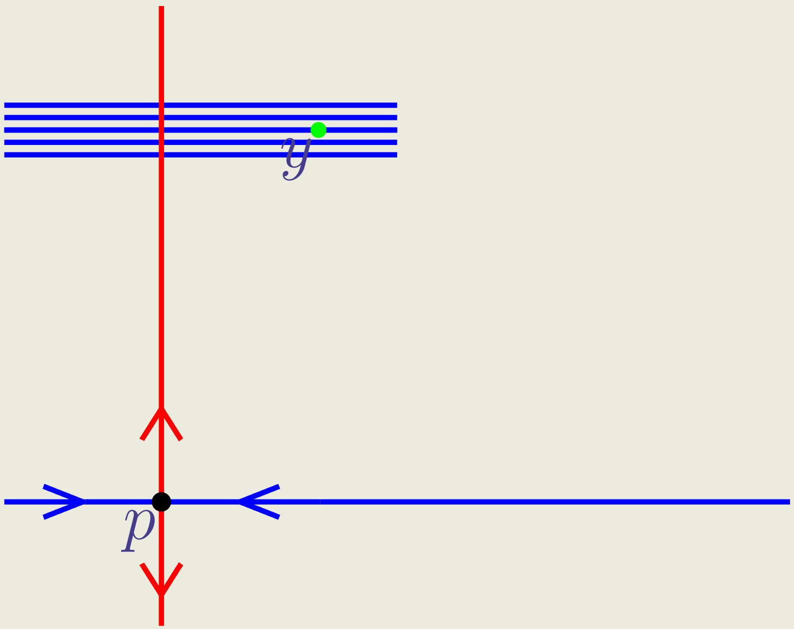

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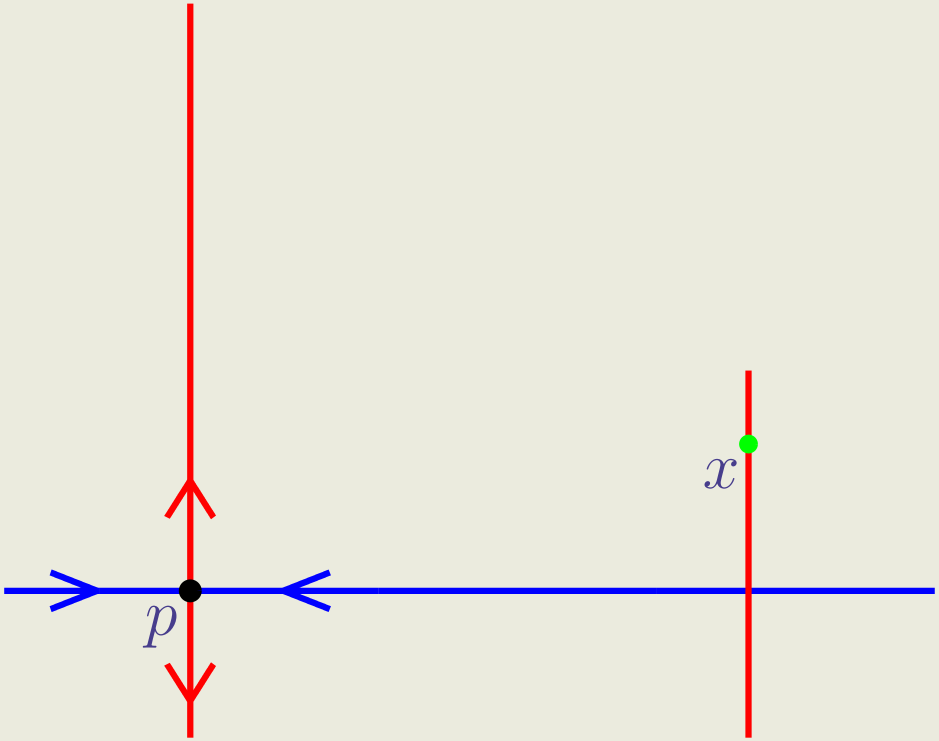
- $\widetilde{W}^s_{loc}(w) \pitchfork W^u(p)$ for all w near y in the Pesin block
- $m(w : *) > 0$
- $w \mapsto \widetilde{W}^s_{loc}(w)$ continuous in the Pesin block

Proof - $B^s(p) \doteq B^u(p)$



- $x \in B^u(p) \cap S_{B^s(p)}$

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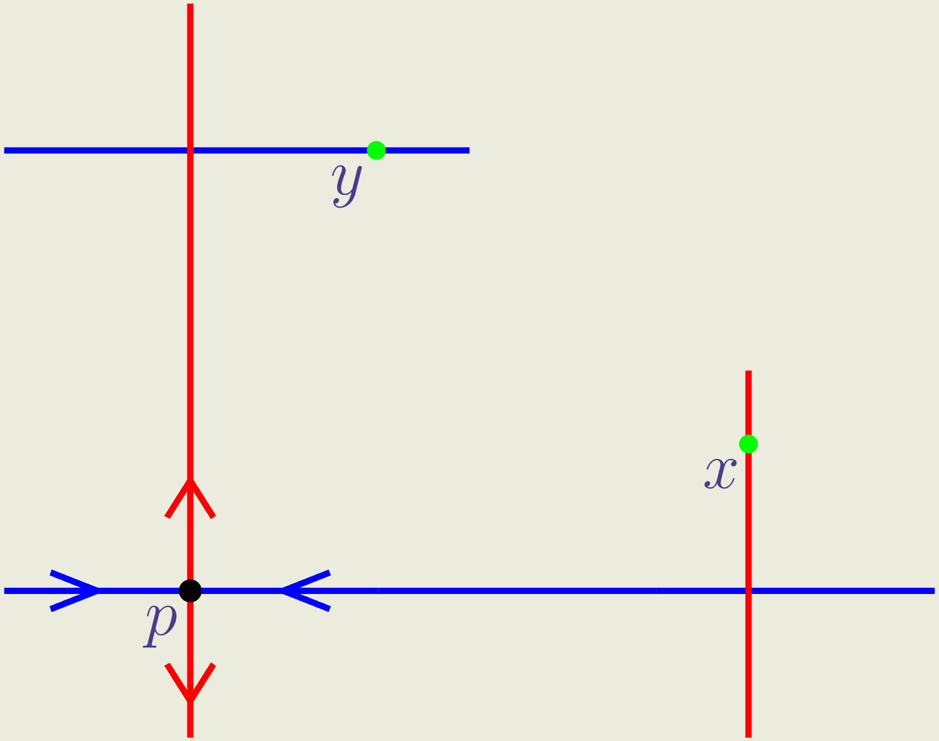

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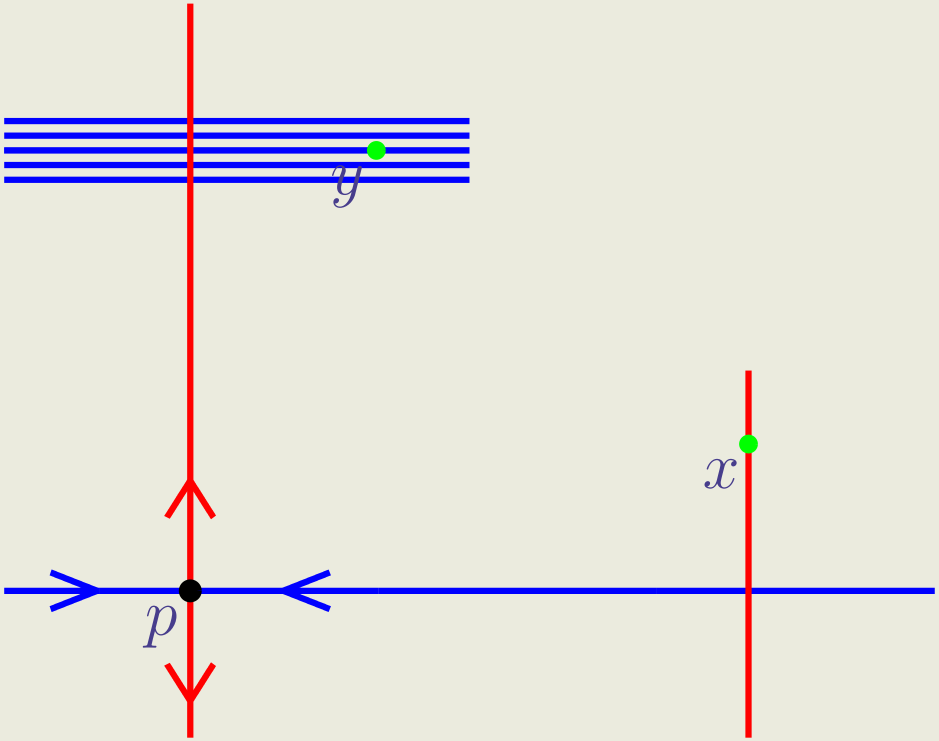
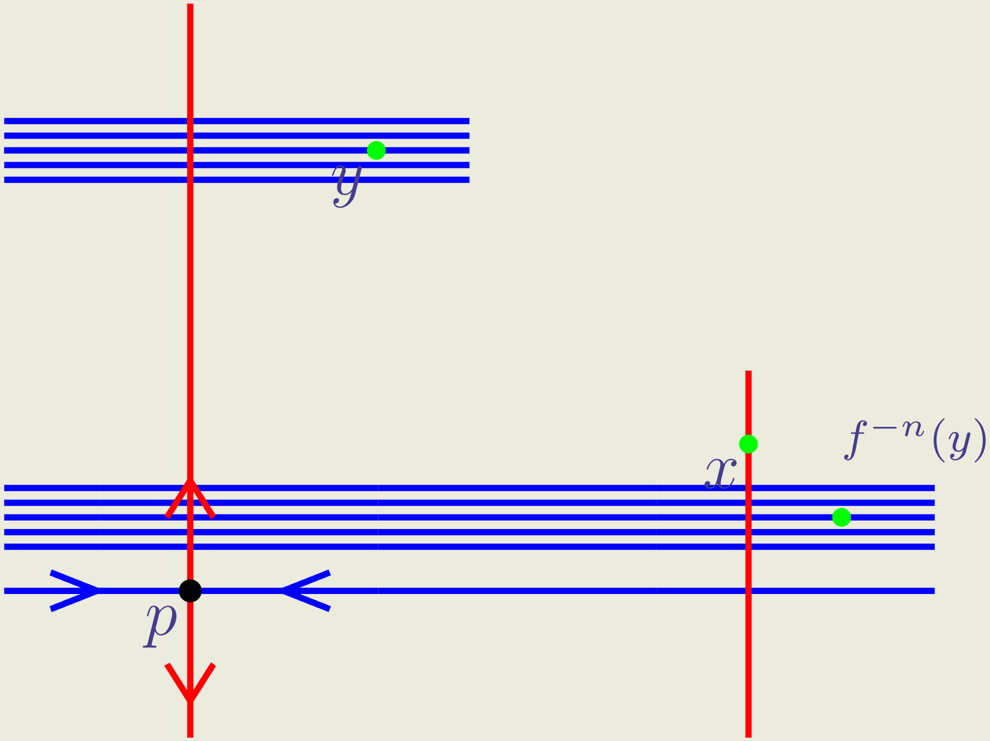


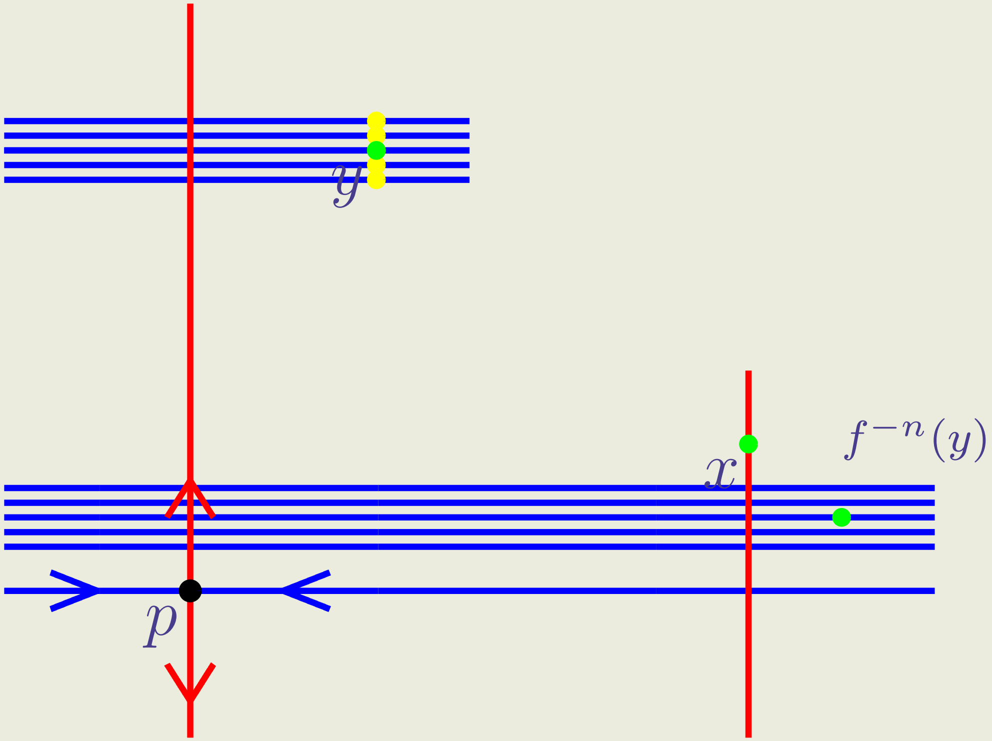
- $w : \widetilde{W}^s_{loc}(w) \pitchfork W^u(p)$
  $m(w : *) > 0$ near y

Proof - $B^s(p) \doteq B^u(p)$



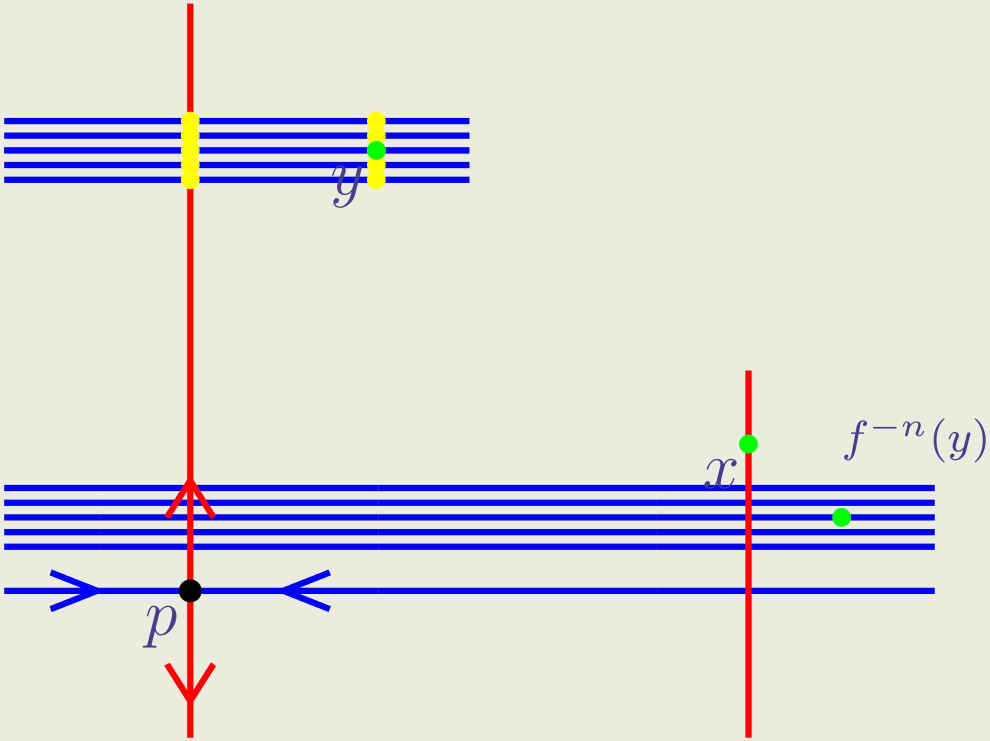
- $\widetilde{W}^s(f^{-n}(y)) \pitchfork \widetilde{W}^u(x)$

Proof - $B^s(p) \doteq B^u(p)$



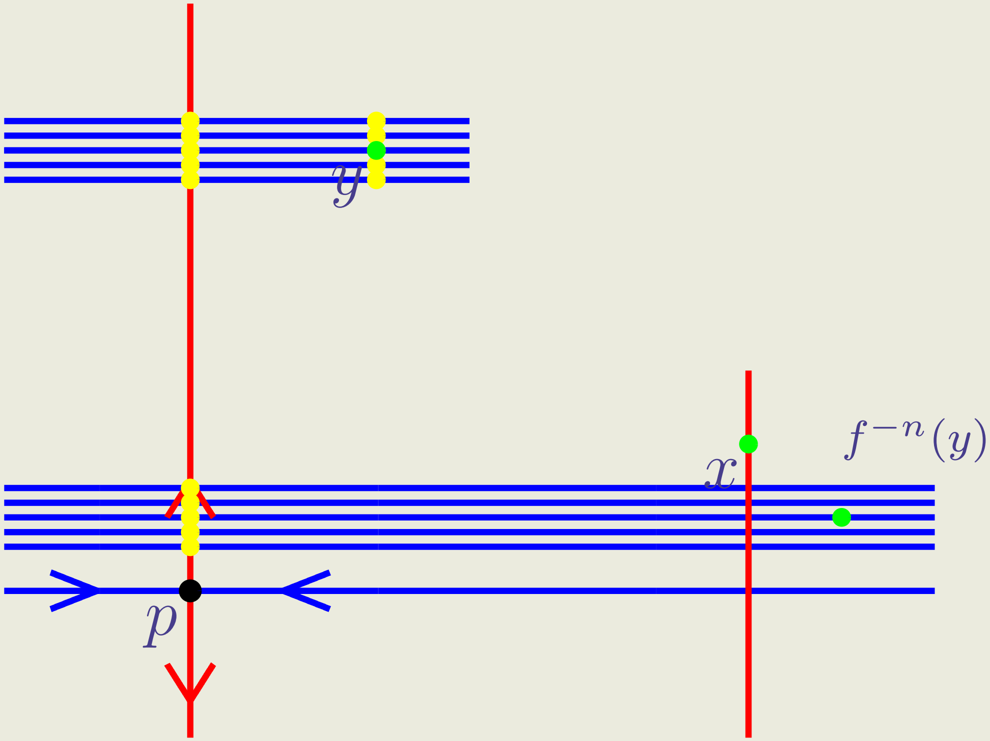
- $m(L) > 0$

Proof - $B^s(p) \doteq B^u(p)$



- $m(L) > 0$
- absolute continuity

Proof - $B^s(p) \doteq B^u(p)$



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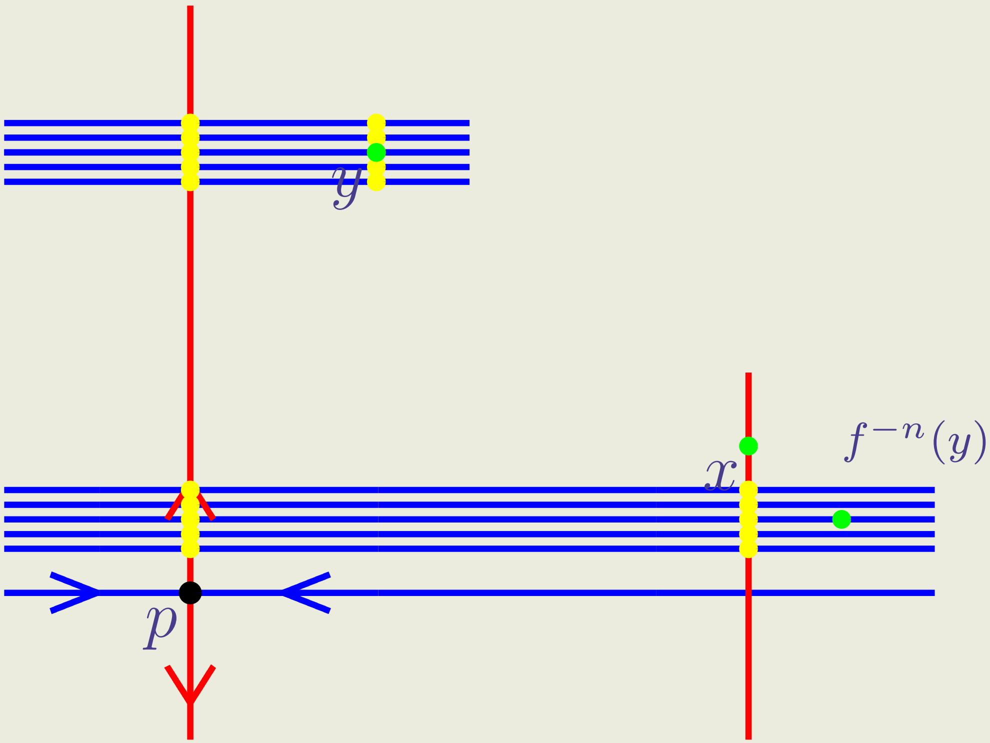
Proof - $B^s(p) \triangleq B^u(p)$



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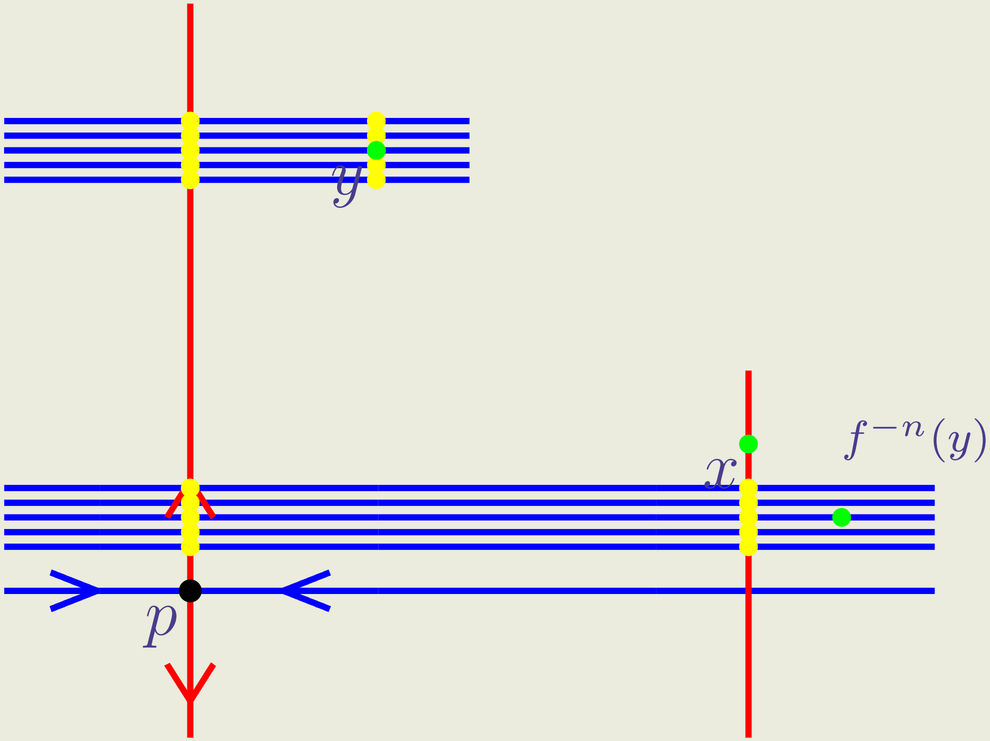

- $m(L) > 0$
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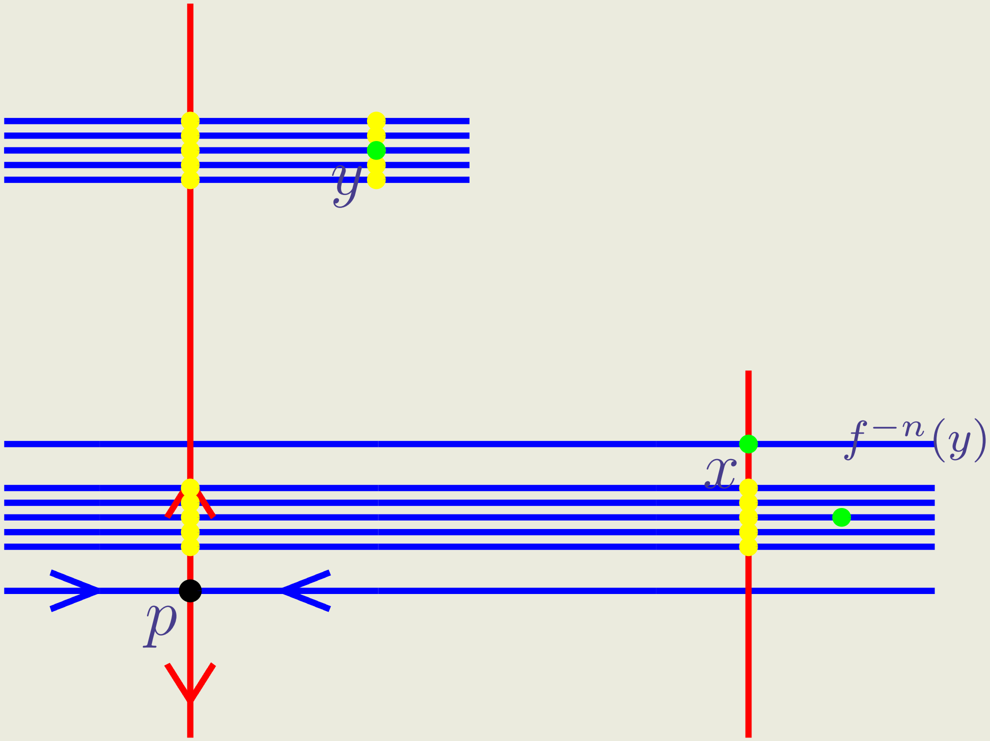
- $m_u(\widetilde{W}^u(x) \cap B^s(p)) > 0$

Proof - $B^s(p) \doteq B^u(p)$



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- $m_u(\widetilde{W}^u(x) \cap B^s(p)) > 0$
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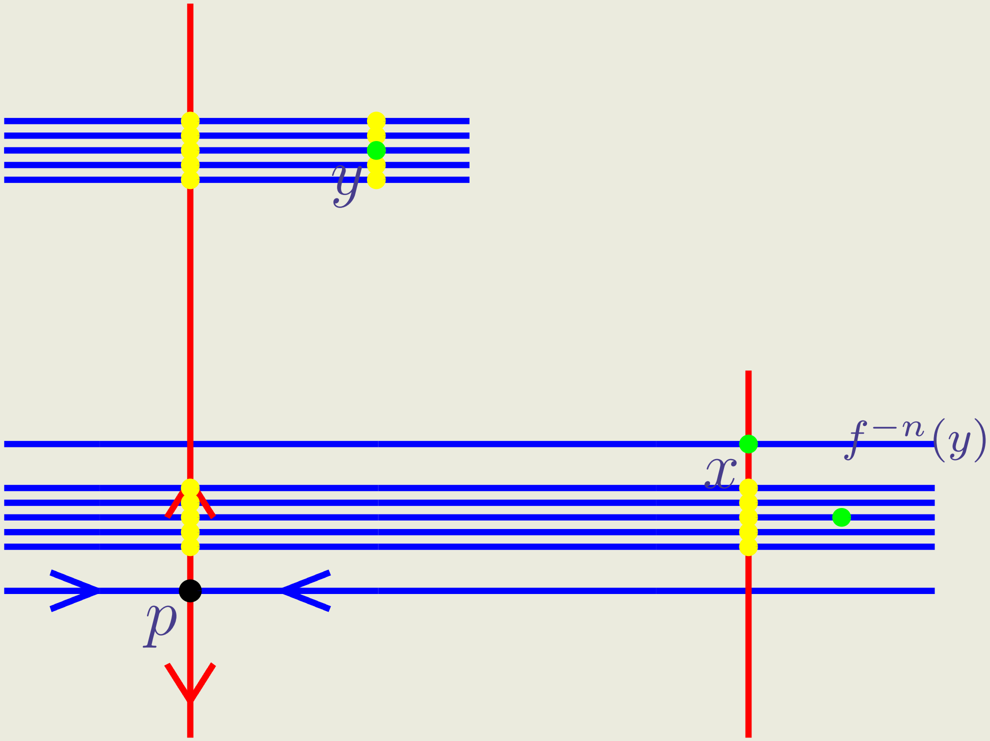
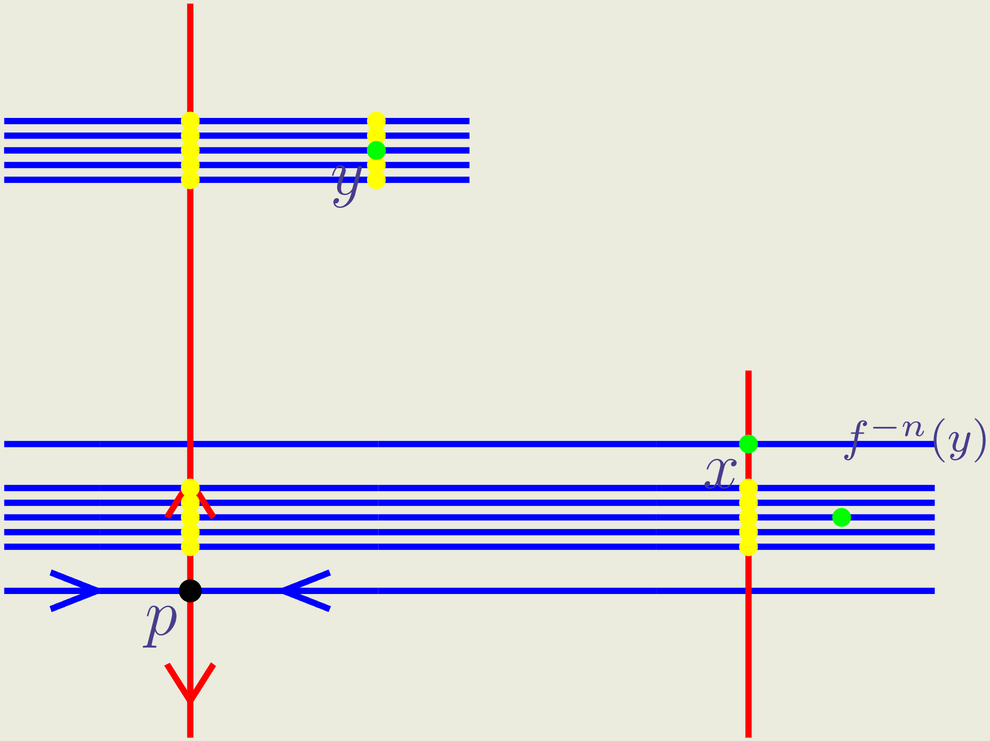
Proof - $B^s(p) \doteq B^u(p)$



- $x \in B^u(p) \cap S_{B^s(p)}$

Proof - $B^s(p) \stackrel{\circ}{=} B^u(p)$



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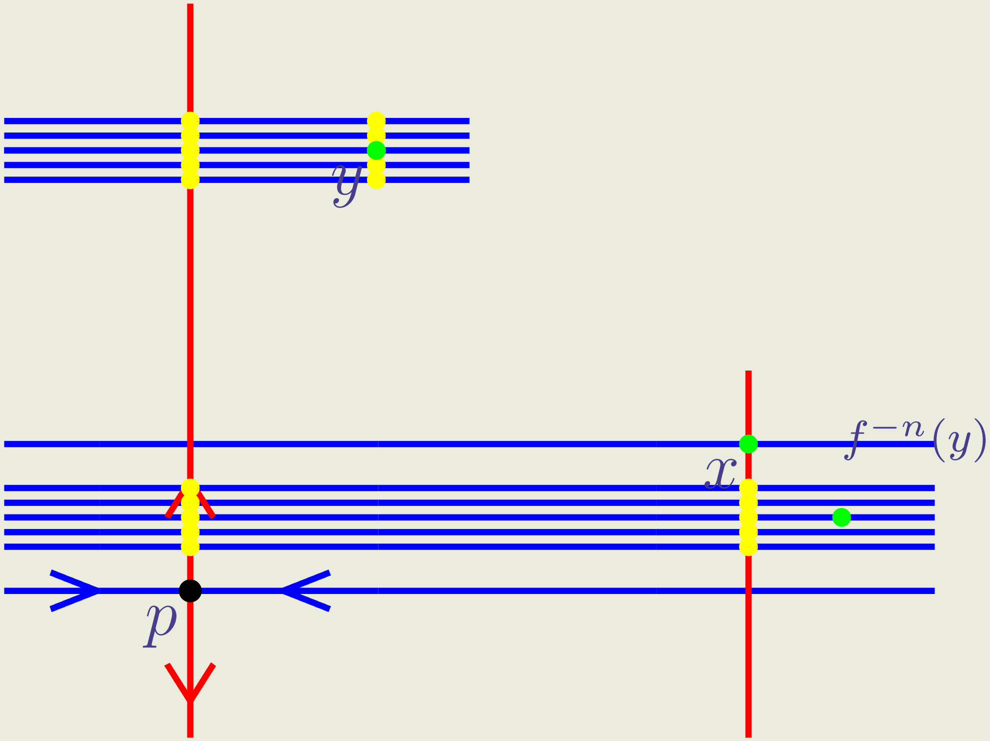

- $x \in B^u(p) \cap S_{B^s(p)}$
- $\Rightarrow x \in B^s(p)$
- $B^u(p) \mathring{\subset} B^s(p)$

Proof - non uniform hyperbolicity

$$s := \dim W^s(p) \qquad u := \dim W^u(p)$$

$$x \in B(p)$$

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- $\widetilde{W}^s(x) \pitchfork W^u(p) \neq \emptyset$

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- $\Rightarrow s(x) \geq s$

Proof - non uniform hyperbolicity

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Λ_1, Λ_2 ergodic components of $B(p)$

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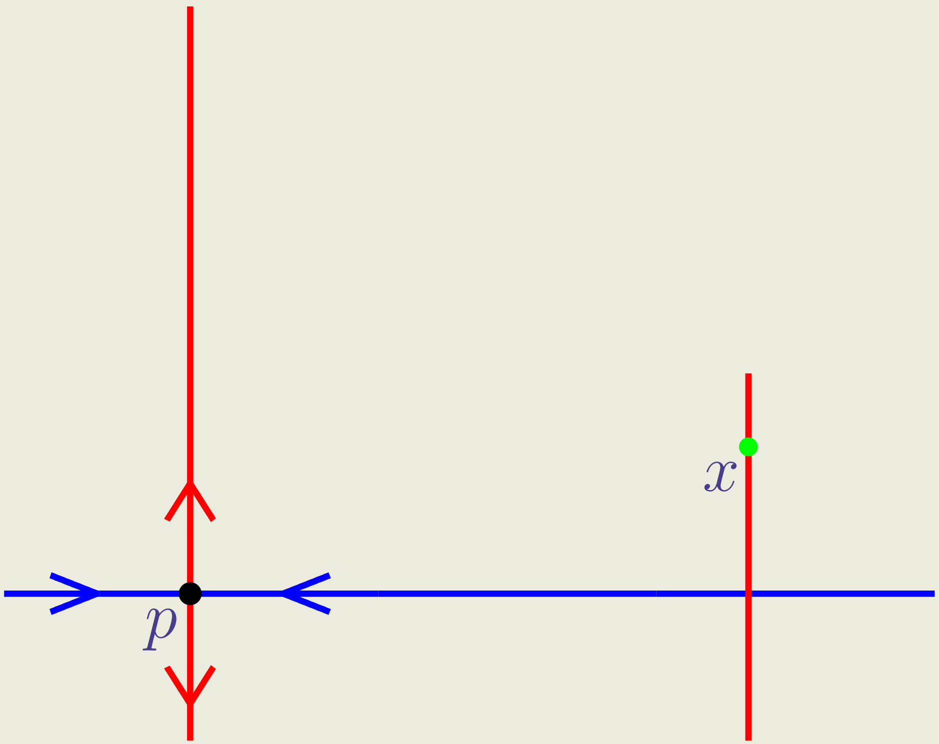

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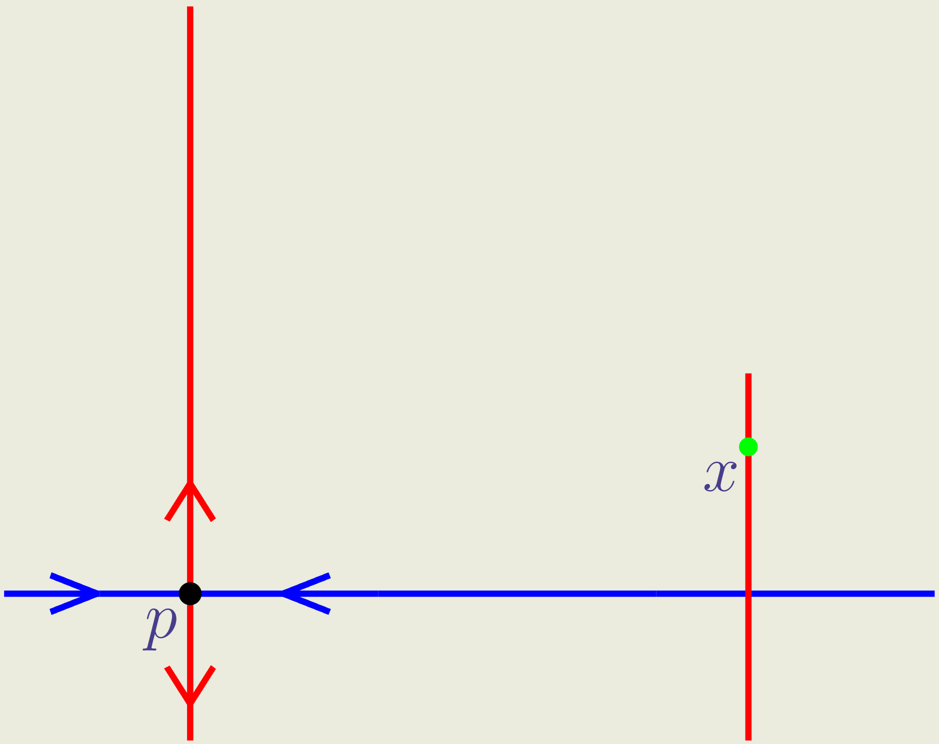

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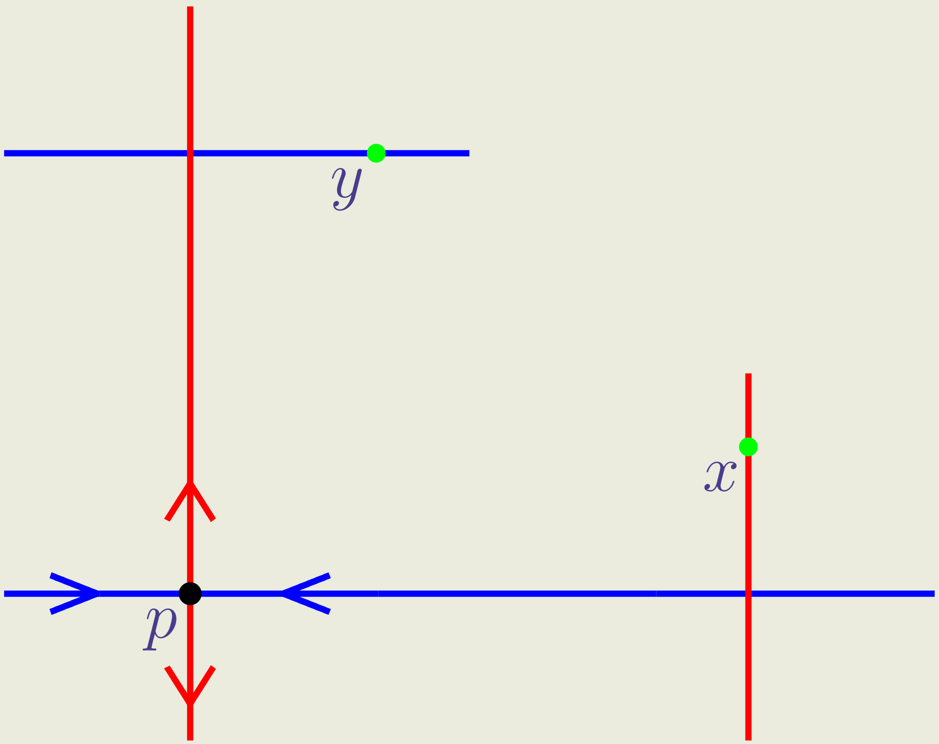

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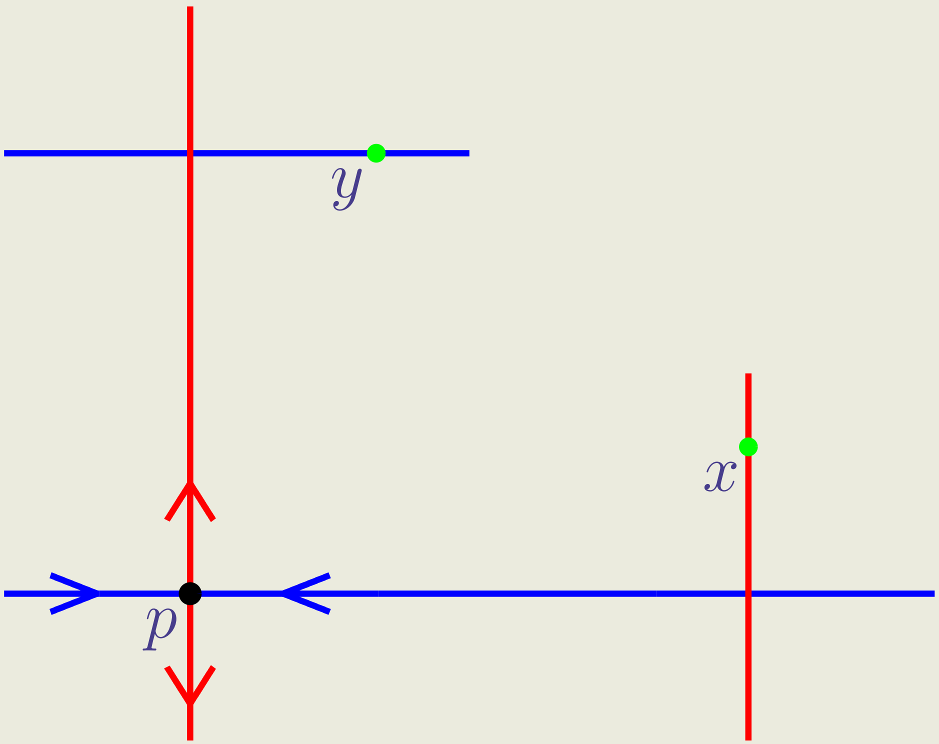

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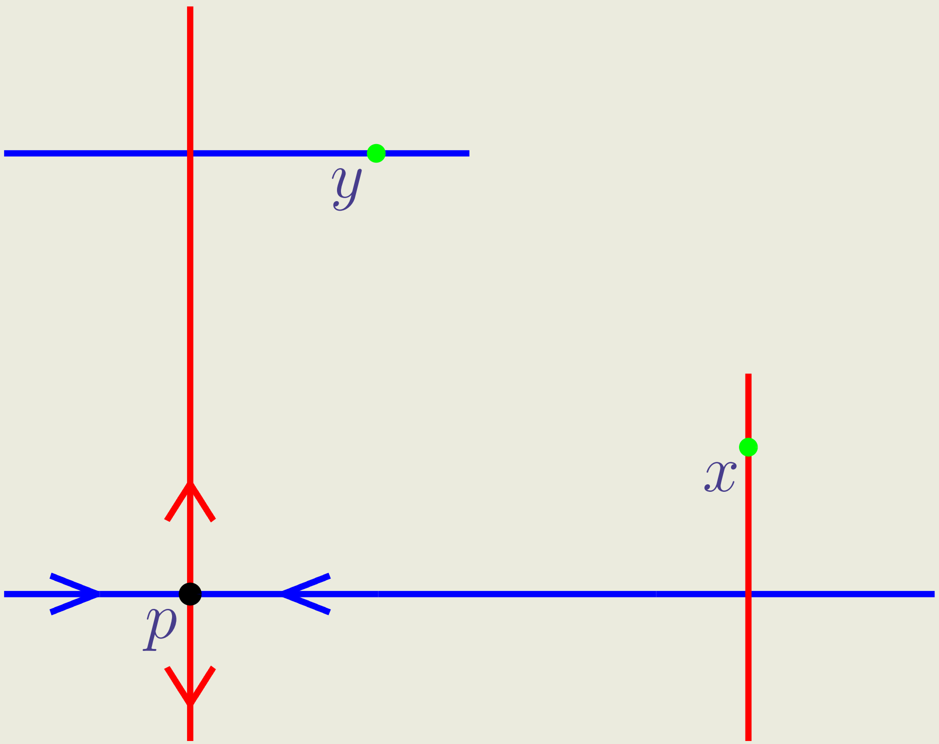

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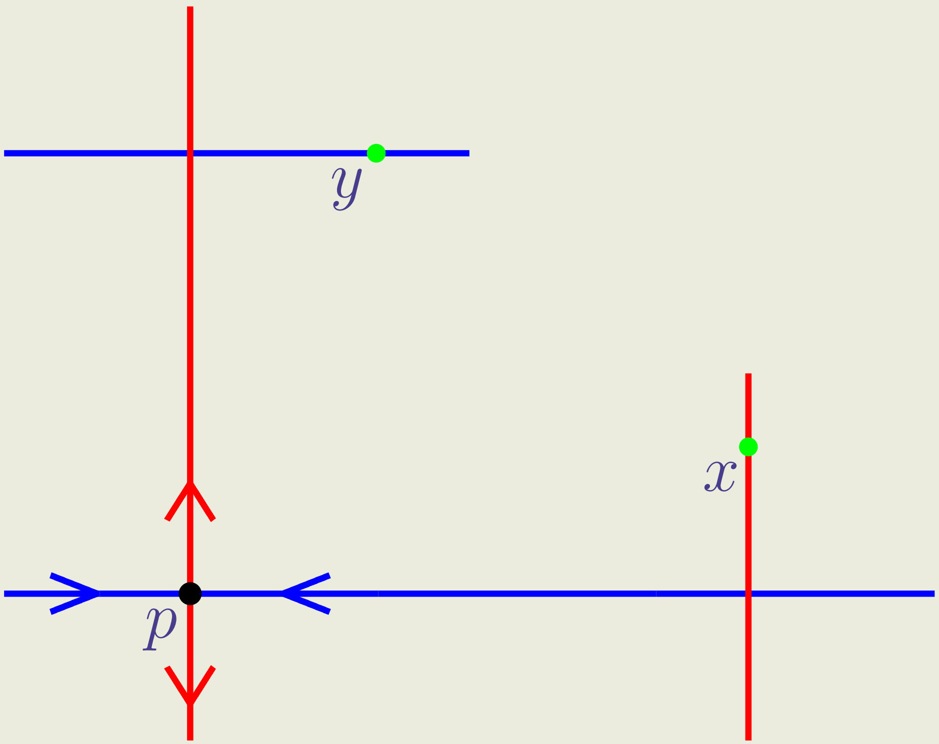

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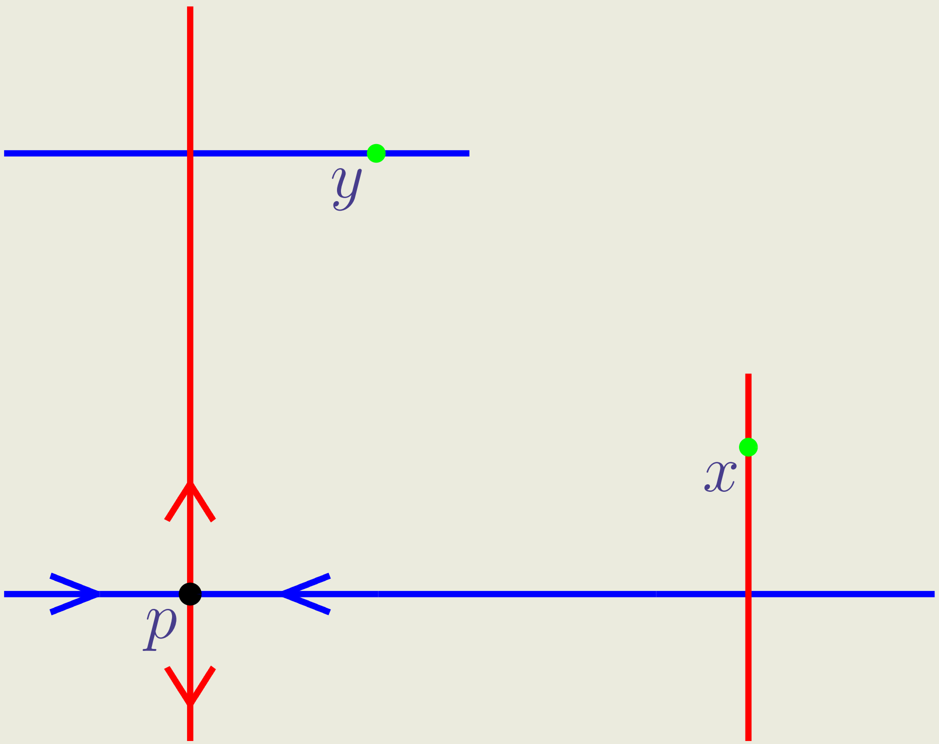

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$\square$

Consequences

- C^1 Pugh-Shub ergodic conjecture for $\dim E^c = 2$

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- $\#\{$SRB measures$\} \leq 1$ for transitive $f : M^2 \to M^2$