Partially hyperbolic dynamics in dimension 3

Jana Rodriguez Hertz

Universidad de la República
Uruguay

Global Dynamics beyond Uniform Hyperbolicity
Luminy 2011

setting

- M^3 closed Riemannian 3-manifold

setting

- M^3 closed Riemannian 3-manifold
- $f : M \to M$ partially hyperbolic diffeomorphism

setting

- M^3 closed Riemannian 3-manifold
- $f : M \to M$ partially hyperbolic diffeomorphism
- f conservative (not always, but most of the time)

partial hyperbolicity

Definition

$f : M^3 \to M^3$ is partially hyperbolic

$$TM = E^s \oplus E^c \oplus E^u$$

partial hyperbolicity

Definition

$f : M^3 \to M^3$ is partially hyperbolic

$$TM = E^s \oplus E^c \oplus E^u$$

E^s ↑ contracting

E^c ↑ intermediate

E^u ↑ expanding

partial hyperbolicity

Definition

$f : M^3 \to M^3$ is partially hyperbolic

$$TM = \underset{\substack{\uparrow \\ \text{1-dim}}}{E^s} \oplus \underset{\substack{\uparrow \\ \text{1-dim}}}{E^c} \oplus \underset{\substack{\uparrow \\ \text{1-dim}}}{E^u}$$

example

example

$$f : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1$$

example

example

$$f : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1$$

such that

$$f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times id$$

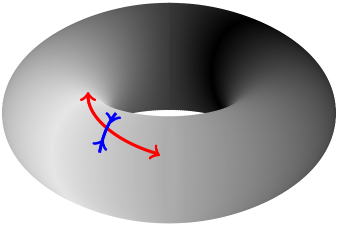
example

example

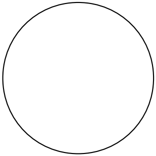
$$f : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1$$

such that

$$f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times id$$



$\times$



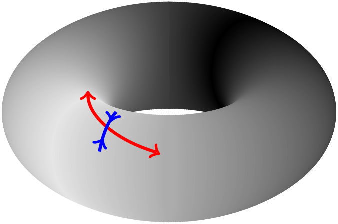
example

example

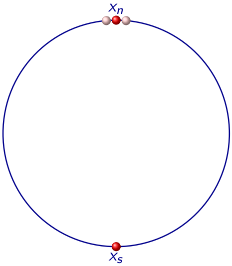
$$f : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1$$

such that

$$f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times NPSP$$



$\times$



remark

- in both examples, there is a foliation tangent to $E^s \oplus E^u$.

remark

- in both examples, there is a foliation tangent to $E^s \oplus E^u$.
- However, there are C^∞ perturbations such that

remark

- in both examples, there is a foliation tangent to $E^s \oplus E^u$.
- However, there are C^∞ perturbations such that
- there is no leaf tangent to $E^s \oplus E^u$ [HHU–08], [BHHTU–08]

some problems

- ergodicity of ph diffeomorphisms

some problems

- ergodicity of ph diffeomorphisms
- dynamical coherence of ph diffeomorphisms

some problems

- ergodicity of ph diffeomorphisms
- dynamical coherence of ph diffeomorphisms
- classification of ph diffeomorphisms

stable ergodicity

Definition (stable ergodicity)

stable ergodicity

Definition (stable ergodicity)

- $f : M \to M$ is stably ergodic if

stable ergodicity

Definition (stable ergodicity)

- $f : M \to M$ is stably ergodic if
- there is $\mathcal{U}^1(f)$ such that

stable ergodicity

Definition (stable ergodicity)

- $f : M \to M$ is stably ergodic if
- there is $\mathcal{U}^1(f)$ such that
- $$\mathcal{U}^1(f) \cap \mathrm{Diff}^2_m(f) \subset \{\text{ergodic}\}$$

frequency of ergodicity

conjecture [Pugh-Shub]

Stable ergodicity is C^r-dense among conservative partially hyperbolic diffeomorphisms

frequency of ergodicity

Theorem (HHU-08)

Stable ergodicity is C^∞-dense among conservative partially hyperbolic diffeomorphisms

question

question

Can we describe precisely the non-ergodic conservative partially hyperbolic diffeomorphisms?

question

question

Can we describe precisely the 3-manifolds supporting non-ergodic conservative partially hyperbolic diffeomorphisms?

question

question

Are there 3-manifolds where all conservative partially hyperbolic diffeomorphisms are ergodic?

question

question

Are there 3-manifolds where all conservative partially hyperbolic diffeomorphisms are ergodic? YES

theorem

Theorem (HHU-08)

If M is a 3-nilmanifold different from the 3-torus,

theorem

Theorem (HHU-08)

If M is a 3-nilmanifold different from the 3-torus, then

- *all conservative partially hyperbolic diffeomorphisms are ergodic*

question

question

Can we describe precisely the 3-manifolds supporting non-ergodic conservative partially hyperbolic diffeomorphisms?

conjecture

conjecture [HHU]

The only 3-manifolds supporting non-ergodic conservative partially hyperbolic diffeomorphisms are:

conjecture

conjecture [HHU]

The only 3-manifolds supporting non-ergodic conservative partially hyperbolic diffeomorphisms are:

- the 3-torus,

conjecture

conjecture [HHU]

The only 3-manifolds supporting non-ergodic conservative partially hyperbolic diffeomorphisms are:

- the 3-torus,
- the mapping torus of $-id : \mathbb{T}^2 \to \mathbb{T}^2$

conjecture

conjecture [HHU]

The only 3-manifolds supporting non-ergodic conservative partially hyperbolic diffeomorphisms are:

- the 3-torus,
- the mapping torus of $-id : \mathbb{T}^2 \to \mathbb{T}^2$
- the mapping tori of hyperbolic automorphisms of $\mathbb{T}^2$

integrability

- $f : M \to M$ partially hyperbolic

integrability

- $f : M \to M$ partially hyperbolic
- $TM = E^s \oplus E^c \oplus E^u$, then

integrability

- $f : M \to M$ partially hyperbolic
- $TM = E^s \oplus E^c \oplus E^u$, then
- $\exists!$ foliation $\mathcal{F}^s$ tangent to E^s

integrability

- $f : M \to M$ partially hyperbolic
- $TM = E^s \oplus E^c \oplus E^u$, then
- $\exists!$ foliation $\mathcal{F}^s$ tangent to E^s
- $\exists!$ foliation $\mathcal{F}^u$ tangent to E^u

integrability

- $f : M \to M$ partially hyperbolic
- $TM = E^s \oplus E^c \oplus E^u$, then
- $\exists!$ foliation $\mathcal{F}^s$ tangent to E^s
- $\exists!$ foliation $\mathcal{F}^u$ tangent to E^u
- ($\mathcal{F}^s$ and $\mathcal{F}^u$ are invariant)

dynamical coherence

Definition (dynamical coherence)

dynamical coherence

Definition (dynamical coherence)

- $f : M \to M$ partially hyperbolic

dynamical coherence

Definition (dynamical coherence)

- $f : M \to M$ partially hyperbolic
- is dynamically coherent if

dynamical coherence

Definition (dynamical coherence)

- $f : M \to M$ partially hyperbolic
- is dynamically coherent if
  1. $\exists$ invariant foliation $\mathcal{F}^{cs}$ tangent to $E^s \oplus E^c$

dynamical coherence

Definition (dynamical coherence)

- $f : M \to M$ partially hyperbolic
- is dynamically coherent if
  1. $\exists$ invariant foliation $\mathcal{F}^{cs}$ tangent to $E^s \oplus E^c$
  2. $\exists$ invariant foliation $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$

remark

remark

remark

remark

- If f is dynamically coherent, then

remark

remark

- If f is dynamically coherent, then
- $\exists$ invariant foliation $\mathcal{F}^c$

remark

remark

- If f is dynamically coherent, then
- $\exists$ invariant foliation $\mathcal{F}^c$
- that sub-foliates $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$

theorem

Theorem (Brin-Burago-Ivanov08)

If $f : \mathbb{T}^3 \to \mathbb{T}^3$ is absolutely partially hyperbolic, then f is dynamically coherent.

question

question

If $f : M^3 \to M^3$ is partially hyperbolic, is f dynamically coherent?

question

question

If $f : M^3 \to M^3$ is partially hyperbolic, is f dynamically coherent?
NO

theorem

Theorem (HHU09)

There exists an open set $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^3)$

theorem

Theorem (HHU09)

There exists an open set $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^3)$ *such that all* $f \in \mathcal{U}$

theorem

Theorem (HHU09)

There exists an open set $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^3)$ *such that all* $f \in \mathcal{U}$

- *are partially hyperbolic*

theorem

Theorem (HHU09)

There exists an open set $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^3)$ *such that all* $f \in \mathcal{U}$

- *are partially hyperbolic*
- *are not dynamically coherent*

conjecture

conjecture [HHU]

All conservative partially hyperbolic diffeomorphisms of a 3-manifold are dynamically coherent.

examples of ph dynamics

known ph dynamics in dimension 3

examples of ph dynamics

known ph dynamics in dimension 3

- perturbations of time-one maps of Anosov flows

examples of ph dynamics

known ph dynamics in dimension 3

- perturbations of time-one maps of Anosov flows
- certain skew-products

examples of ph dynamics

known ph dynamics in dimension 3

- perturbations of time-one maps of Anosov flows
- certain skew-products
- certain DA-maps

question

question

are there more examples?

conjecture Pujals

conjecture [Pujals01]

If $f : M^3 \to M^3$ is a transitive partially hyperbolic diffeomorphism, then f is (finitely covered by) either

conjecture Pujals

conjecture [Pujals01]

If $f : M^3 \to M^3$ is a transitive partially hyperbolic diffeomorphism, then f is (finitely covered by) either

1. a perturbation of a time-one map of an Anosov flow

conjecture Pujals

conjecture [Pujals01]

If $f : M^3 \to M^3$ is a transitive partially hyperbolic diffeomorphism, then f is (finitely covered by) either

1. a perturbation of a time-one map of an Anosov flow
2. a skew-product

conjecture Pujals

conjecture [Pujals01]

If $f : M^3 \to M^3$ is a transitive partially hyperbolic diffeomorphism, then f is (finitely covered by) either

1. a perturbation of a time-one map of an Anosov flow
2. a skew-product
3. a DA-map

theorem

Theorem (Bonatti-Wilkinson04)

$f : M^3 \to M^3$ transitive and partially hyperbolic, such that

theorem

Theorem (Bonatti-Wilkinson04)

$f : M^3 \to M^3$ transitive and partially hyperbolic, such that

- *$\exists \mathcal{C}$ invariant circle*

theorem

Theorem (Bonatti-Wilkinson04)

$f : M^3 \to M^3$ transitive and partially hyperbolic, such that

- *$\exists \mathcal{C}$ invariant circle*
- *$\exists R > 0$ such that $W^s_R(\mathcal{C}) \cap W^u_R(\mathcal{C}) \setminus \mathcal{C}$ contains a a circle*

theorem

Theorem (Bonatti-Wilkinson04)

$f : M^3 \to M^3$ transitive and partially hyperbolic, such that

- *$\exists \mathcal{C}$ invariant circle*
- *$\exists R > 0$ such that $W^s_R(\mathcal{C}) \cap W^u_R(\mathcal{C}) \setminus \mathcal{C}$ contains a a circle*

then

theorem

Theorem (Bonatti-Wilkinson04)

$f : M^3 \to M^3$ transitive and partially hyperbolic, such that

- *$\exists \mathcal{C}$ invariant circle*
- *$\exists R > 0$ such that $W^s_R(\mathcal{C}) \cap W^u_R(\mathcal{C}) \setminus \mathcal{C}$ contains a a circle*

then

1. *f is dynamically coherent*

theorem

Theorem (Bonatti-Wilkinson04)

$f : M^3 \to M^3$ transitive and partially hyperbolic, such that

- *$\exists \mathcal{C}$ invariant circle*
- *$\exists R > 0$ such that $W^s_R(\mathcal{C}) \cap W^u_R(\mathcal{C}) \setminus \mathcal{C}$ contains a a circle*

then

1. *f is dynamically coherent*
2. *f is finitely covered by a skew-product*

conjecture

conjeture 1 [HHU]

If $f : M^3 \to M^3$ is partially hyperbolic and dynamically coherent, then either

conjecture

conjeture 1 [HHU]

If $f : M^3 \to M^3$ is partially hyperbolic and dynamically coherent, then either

1. f^n is a perturbation of a time-one map of an Anosov flow,

conjecture

conjeture 1 [HHU]

If $f : M^3 \to M^3$ is partially hyperbolic and dynamically coherent, then either

1. f^n is a perturbation of a time-one map of an Anosov flow,
2. f is a skew-product, or

conjecture

conjeture 1 [HHU]

If $f : M^3 \to M^3$ is partially hyperbolic and dynamically coherent, then either

1. f^n is a perturbation of a time-one map of an Anosov flow,
2. f is a skew-product, or
3. f is a DA-map

conjecture

conjeture 1 [HHU]

If $f : M^3 \to M^3$ is partially hyperbolic and dynamically coherent, then either

1. f^n is a perturbation of a time-one map of an Anosov flow,
2. f is a skew-product, or
3. f is a DA-map

Moreover, in case:

1. f^n is leafwise conjugate to an Anosov flow

conjecture

conjeture 1 [HHU]

If $f : M^3 \to M^3$ is partially hyperbolic and dynamically coherent, then either

1. f^n is a perturbation of a time-one map of an Anosov flow,
2. f is a skew-product, or
3. f is a DA-map

Moreover, in case:

1. f^n is leafwise conjugate to an Anosov flow
2. f is leafwise conjugate to a skew-product with linear base

conjecture

conjeture 1 [HHU]

If $f : M^3 \to M^3$ is partially hyperbolic and dynamically coherent, then either

1. f^n is a perturbation of a time-one map of an Anosov flow,
2. f is a skew-product, or
3. f is a DA-map

Moreover, in case:

1. f^n is leafwise conjugate to an Anosov flow
2. f is leafwise conjugate to a skew-product with linear base
3. f is leafwise conjugate to an Anosov map in $\mathbb{T}^3$.

theorem 1

Theorem (Hammerlindl09)

If $f : \mathbb{T}^3 \to \mathbb{T}^3$ is absolutely partially hyperbolic,

theorem 1

Theorem (Hammerlindl09)

If $f : \mathbb{T}^3 \to \mathbb{T}^3$ is absolutely partially hyperbolic, then f is leafwise conjugate to a ph linear map

theorem 2

Theorem (Hammerlind10)

If $f : N^3 \to N^3$ is absolutely partially hyperbolic, N nilmanifold,

theorem 2

Theorem (Hammerlind10)

If $f : N^3 \to N^3$ is absolutely partially hyperbolic, N nilmanifold, then f is conjugate to a skew-product

conjecture 2

conjecture 2 [HHU]

If $f : M^3 \to M^3$ is not dynamically coherent,

conjecture 2

conjecture 2 [HHU]

If $f : M^3 \to M^3$ is not dynamically coherent, then

- there exists an (attracting) invariant torus tangent to $E^c \oplus E^u$, or

conjecture 2

conjecture 2 [HHU]

If $f : M^3 \to M^3$ is not dynamically coherent, then

- there exists an (attracting) invariant torus tangent to $E^c \oplus E^u$, or
- there exists a (repelling) invariant torus tangent to $E^s \oplus E^c$.