Partially hyperbolic dynamics in dimension 3

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Global Dynamics beyond Uniform Hyperbolicity
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partial hyperbolicity

Definition (partially hyperbolic)

$f : M \to M$ is partially hyperbolic if

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problems

- ergodicity

problems

- ergodicity
- dynamical coherence

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- classification

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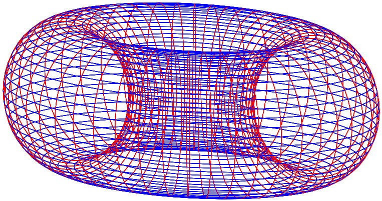
$\Bigg\} \rightarrow$ **Anosov torus**

Anosov torus

Anosov torus

- T embedded 2-torus



Anosov torus

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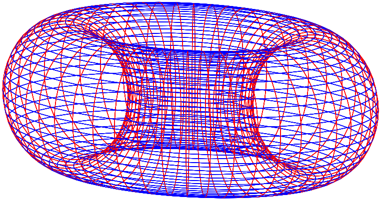

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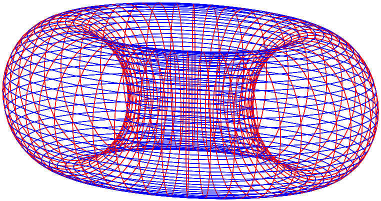

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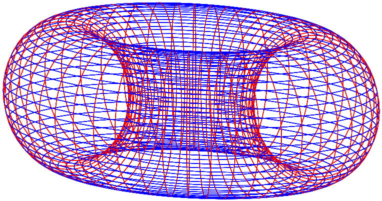
  1. $f(T) = T$



Anosov torus

Anosov torus

- T embedded 2-torus
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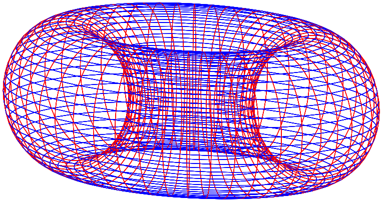
- if $\exists f : M \to M$ s.t.
  1. $f(T) = T$
  2. $f|_T$ isotopic to Anosov



theorem

Theorem (HHU11)

If an irreducible M contains an Anosov torus,

theorem

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If an irreducible M contains an Anosov torus, *then M is either*

theorem

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1. $\mathbb{T}^3$

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3. *a mapping torus of a hyperbolic automorphism of* $\mathbb{T}^2$

conjecture

conjecture [HHU08]

If $f : M^3 \to M^3$ is a conservative non-ergodic partially hyperbolic diffeomorphism,

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accessibility classes

Definition (accessibility class)

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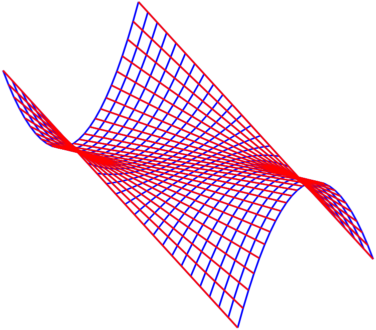

examples

examples

1. $f = \text{Anosov} \times id : \mathbb{T}^3 \to \mathbb{T}^3$

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   - $\Rightarrow$ all accessibility classes are invariant tori

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   - $\Rightarrow$ all accessibility classes are invariant tori
2. $g = \text{Anosov} \times \text{rotation} : \mathbb{T}^3 \to \mathbb{T}^3$
   - $\Rightarrow$ all accessibility classes are tori (non-invariant)

accessibility

Definition (accessibility)

$f : M \to M$ has the accessibility property

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Definition (accessibility)

$f : M \to M$ has the accessibility property if
M is an accessibility class

theorem

Theorem (BW10, HHU08)

$f : M^3 \to M^3$ conservative and partially hyperbolic

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- *accessibility $\Rightarrow$ ergodicity*

theorem

Theorem (BW10, HHU08)

$f : M^3 \to M^3$ conservative and partially hyperbolic

- *accessibility $\Rightarrow$ essential accessibility $\Rightarrow$ ergodicity*

problem

$$\{\text{non ergodic}\} \subset \{\text{non accessible}\}$$

accessibility classes

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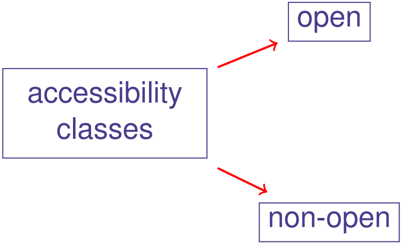

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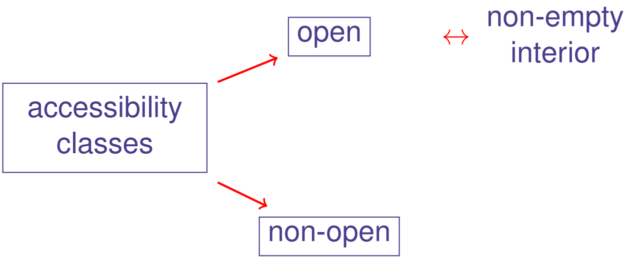

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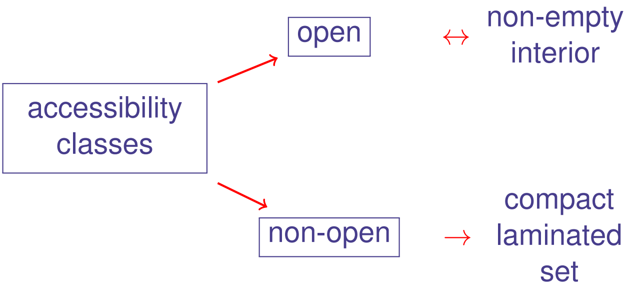

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then M is either:

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3. *a mapping torus of a hyperbolic automorphism of $\mathbb{T}^2$*

theorem 2

Theorem (HHU08)

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If $f : M^3 \to M^3$ conservative partially hyperbolic, then, either:

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2. *there exists an invariant lamination by accessibility classes, extensible to a foliation without compact leaves*

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If $f : M^3 \to M^3$ conservative partially hyperbolic, then, either:

1. *there exists a compact accessibility class*
2. *there exists an invariant lamination by accessibility classes, extensible to a foliation without compact leaves*
3. *there exists a* minimal *foliation by accessibility classes*

remark

In the case (2), each boundary leaf of the lamination:

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- is a periodic accessibility class

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- is a periodic accessibility class
- periodic points are dense w.r.t. the intrinsic topology
- stable and unstable manifolds of periodic points are dense w.r.t. the intrinsic topology

question

question

Can you find an example of a conservative partially hyperbolic diffeo:

question

question

Can you find an example of a conservative partially hyperbolic diffeo:

- non-ergodic

question

question

Can you find an example of a conservative partially hyperbolic diffeo:

- non-ergodic
- with no compact accessibility classes?

dynamical coherence

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- is dynamically coherent if
  1. $\exists$ invariant foliation $\mathcal{F}^{cs}$ tangent to $E^s \oplus E^c$

dynamical coherence

Definition (dynamical coherence)

- $f : M \to M$ partially hyperbolic
- is dynamically coherent if
  1. $\exists$ invariant foliation $\mathcal{F}^{cs}$ tangent to $E^s \oplus E^c$
  2. $\exists$ invariant foliation $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$

remark

remark

remark

remark

- If f is dynamically coherent, then

remark

remark

- If f is dynamically coherent, then
- $\exists$ invariant foliation $\mathcal{F}^c$

remark

remark

- If f is dynamically coherent, then
- $\exists$ invariant foliation $\mathcal{F}^c$
- that sub-foliates $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$

question

question

If $f : M^3 \to M^3$ is partially hyperbolic, is f dynamically coherent?

theorem

Theorem (Brin-Burago-Ivanov08)

If $f : \mathbb{T}^3 \to \mathbb{T}^3$ *is absolutely partially hyperbolic,*

theorem

Theorem (Brin-Burago-Ivanov08)

If $f : \mathbb{T}^3 \to \mathbb{T}^3$ is absolutely partially hyperbolic, then

- *$\mathcal{F}^s$ and $\mathcal{F}^u$ are quasi-isometric*

theorem

Theorem (Brin-Burago-Ivanov08)

If $f : \mathbb{T}^3 \to \mathbb{T}^3$ is absolutely partially hyperbolic, then

- *$\mathcal{F}^s$ and $\mathcal{F}^u$ are quasi-isometric*
- *$\Rightarrow$ f is uniquely dynamically coherent*

theorem

Theorem (Brin-Burago-Ivanov08)

If $f : \mathbb{T}^3 \to \mathbb{T}^3$ is absolutely partially hyperbolic, then

- *$\mathcal{F}^s$ and $\mathcal{F}^u$ are quasi-isometric*
- *$\Rightarrow$ f is uniquely dynamically coherent*

Theorem (Parwani10)

Extended the result to 3-nilmanifolds

theorem

Theorem (HHU10)

If $f : M^3 \to M^3$ is partially hyperbolic, then

theorem

Theorem (HHU10)

If $f : M^3 \to M^3$ is partially hyperbolic, then

- *any invariant $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$*

theorem

Theorem (HHU10)

If $f : M^3 \to M^3$ is partially hyperbolic, then

- *any invariant $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$*
- *cannot have compact leaves*

remarks

remarks

1. If T is a compact leaf tangent to $E^c \oplus E^u$, then T is a torus

remarks

remarks

1. If T is a compact leaf tangent to $E^c \oplus E^u$, then T is a torus
2. If T is invariant, then T is an Anosov torus

example

example [HHU09]

There exists an open set $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^3)$

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There exists an open set $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^3)$
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- are partially hyperbolic

example

example [HHU09]

There exists an open set $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^3)$ such that all $f \in \mathcal{U}$

- are partially hyperbolic
- are not dynamically coherent

conjecture

conjecture [HHU10]

If $f : M^3 \to M^3$ is partially hyperbolic,

conjecture

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If $f : M^3 \to M^3$ is partially hyperbolic, and

- f is not dynamically coherent

conjecture

conjecture [HHU10]

If $f : M^3 \to M^3$ is partially hyperbolic, and

- f is not dynamically coherent

then either:

1. exists an Anosov torus tangent to $E^s \oplus E^c$, or

conjecture

conjecture [HHU10]

If $f : M^3 \to M^3$ is partially hyperbolic, and

- f is not dynamically coherent

then either:

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conjecture

in particular

If M is not:

conjecture

in particular

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conjecture

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If M is not:

1. $\mathbb{T}^3$
2. the mapping torus of $-id$

conjecture

in particular

If M is not:

1. $\mathbb{T}^3$
2. the mapping torus of $-id$
3. the mapping torus of a hyperbolic linear automorphism

then f is dynamically coherent

conjecture 2

conjecture 2 [HHU09]

If $f : M^3 \to M^3$ is conservative, then:

conjecture 2

conjecture 2 [HHU09]

If $f : M^3 \to M^3$ is conservative, then:

- f is dynamically coherent

question

question

Is it true that if $N^3 \neq \mathbb{T}^3$, then

question

question

Is it true that if $N^3 \neq \mathbb{T}^3$, then

all $f : N^3 \to N^3$ ph are dynamically coherent?