Partially hyperbolic dynamics in dimension 3

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Global Dynamics beyond Uniform Hyperbolicity
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partial hyperbolicity

partially hyperbolic

$f : M^3 \to M^3$ is partially hyperbolic

$$TM = E^s \oplus E^c \oplus E^u$$

partial hyperbolicity

partially hyperbolic

$f : M^3 \to M^3$ is partially hyperbolic

$$TM = \underset{\substack{\uparrow \\ \text{contracting}}}{E^s} \oplus \underset{\substack{\uparrow \\ \text{intermediate}}}{E^c} \oplus \underset{\substack{\uparrow \\ \text{expanding}}}{E^u}$$

partial hyperbolicity

partially hyperbolic

$f : M^3 \to M^3$ is partially hyperbolic

$$\begin{array}{ccccccc} TM & = & E^s & \oplus & E^c & \oplus & E^u \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \text{1-dim} & & \text{1-dim} & & \text{1-dim} \end{array}$$

problems

- ergodicity

problems

- ergodicity
- dynamical coherence

problems

- ergodicity
- dynamical coherence
- classification

problems

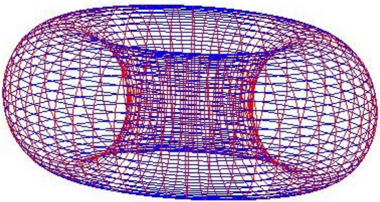
- ergodicity
- dynamical coherence
- classification

$\Bigg\} \to$ **Anosov torus**

Anosov torus

Anosov torus

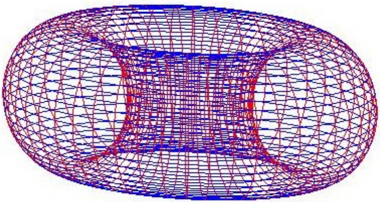
- T embedded 2-torus



Anosov torus

Anosov torus

- T embedded 2-torus
- T Anosov torus



Anosov torus

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- T embedded 2-torus
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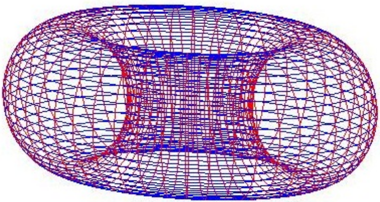
- if $\exists f : M \to M$ s.t.



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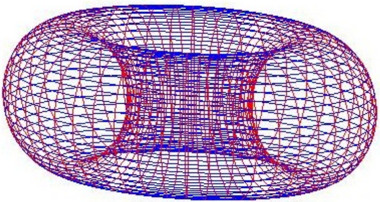
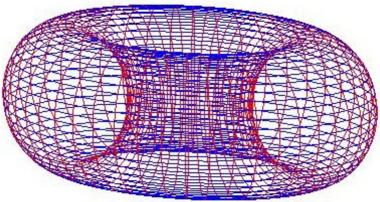
- if $\exists f : M \to M$ s.t.
  1. $f(T) = T$



Anosov torus

Anosov torus

- T embedded 2-torus
- T Anosov torus
- if $\exists f : M \to M$ s.t.
  1. $f(T) = T$
  2. $f|_T$ isotopic to Anosov



remark

lemma

- T Anosov torus

remark

lemma

- T Anosov torus
- $\Rightarrow \exists\, F : M \to M$

remark

lemma

- T Anosov torus
- $\Rightarrow \exists\ F : M \to M$
- $A = F|_T$ hyperbolic automorphism

theorem

theorem (HHU11)

If an irreducible M contains an Anosov torus,

theorem

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If an irreducible M contains an Anosov torus, then M is either

theorem

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1. $\mathbb{T}^3$

theorem

theorem (HHU11)

If an irreducible M contains an Anosov torus, then M is either

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2. the mapping torus of $-id : \mathbb{T}^2 \to \mathbb{T}^2$

theorem

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If an irreducible M contains an Anosov torus, then M is either

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3. a mapping torus of a hyperbolic automorphism of $\mathbb{T}^2$

remark

remark

If $f : M^3 \to M^3$ partially hyperbolic, then

remark

remark

If $f : M^3 \to M^3$ partially hyperbolic, then

- M is irreducible (Burago-Ivanov08, Rosenberg68)

remark

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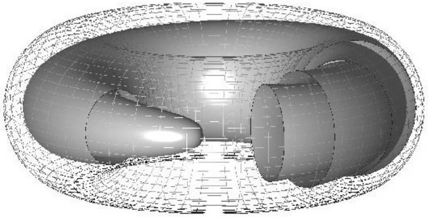
- M is irreducible (Burago-Ivanov08, Rosenberg68)



non-ergodic conjecture

non-ergodic conjecture (HHU)

- $f : M^3 \to M^3$ non-ergodic conservative partially hyperbolic,

non-ergodic conjecture

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harder question

f non-ergodic $\overset{?}{\Rightarrow}$ compact accessibility class

consequences

if this were true

in most manifolds:

consequences

if this were true

in most manifolds:

$$\text{partially hyperbolic} \Rightarrow \text{ergodic}$$

consequences

if this were true

in most manifolds:

partially hyperbolic $\Rightarrow$ ergodic

evidence (HHU08)

true in nilmanifolds $\neq \mathbb{T}^3$

looking for Anosov tori

theorem (HHU08)

If $f : M^3 \to M^3$ conservative partially hyperbolic,

looking for Anosov tori

theorem (HHU08)

If $f : M^3 \to M^3$ conservative partially hyperbolic, then, either:

1. there exists a compact accessibility class

looking for Anosov tori

theorem (HHU08)

If $f : M^3 \to M^3$ conservative partially hyperbolic, then, either:

1. there exists a compact accessibility class
2. there exists an invariant lamination by accessibility classes, extensible to a foliation without compact leaves

looking for Anosov tori

theorem (HHU08)

If $f: M^3 \to M^3$ conservative partially hyperbolic, then, either:

1. there exists a compact accessibility class
2. there exists an invariant lamination by accessibility classes, extensible to a foliation without compact leaves
3. there exists a minimal foliation by accessibility classes

looking for Anosov tori

case ①

$\exists$ compact accessibility class

looking for Anosov tori

case (1)

$\exists$ compact accessibility class

- $\Rightarrow$ ergodicity or Anosov torus

looking for Anosov tori

case ②

invariant lamination

looking for Anosov tori

case (2)

invariant lamination
boundary leaves with intrinsic topology

looking for Anosov tori

case (2)

invariant lamination
boundary leaves with intrinsic topology

- have dense periodic points

looking for Anosov tori

case (2)

invariant lamination
boundary leaves with intrinsic topology

- have dense periodic points
- stable and unstable leaves of periodic points are dense

looking for Anosov tori

case 2

invariant lamination
boundary leaves with intrinsic topology

- have dense periodic points
- stable and unstable leaves of periodic points are dense
- are complete

looking for Anosov tori

case (2)

invariant lamination
boundary leaves with intrinsic topology

- have dense periodic points
- stable and unstable leaves of periodic points are dense
- are complete
- are the boundary leaves Anosov tori?

dynamical coherence

dynamical coherence

dynamical coherence

dynamical coherence

- $f : M \to M$ partially hyperbolic

dynamical coherence

dynamical coherence

- $f : M \to M$ partially hyperbolic
- is dynamically coherent if

dynamical coherence

dynamical coherence

- $f : M \to M$ partially hyperbolic
- is dynamically coherent if
  1. $\exists$ invariant foliation $\mathcal{F}^{cs}$ tangent to $E^s \oplus E^c$

dynamical coherence

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dynamical coherence

dynamical coherence

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- is dynamically coherent if
  1. $\exists$ invariant foliation $\mathcal{F}^{cs}$ tangent to $E^s \oplus E^c$
  2. $\exists$ invariant foliation $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$
  3. $\Rightarrow \exists$ invariant foliation $\mathcal{F}^c$ tangent to E^c

question

question

If $f : M^3 \to M^3$ is partially hyperbolic, is f dynamically coherent?

question

question

If $f : M^3 \to M^3$ is partially hyperbolic, is f dynamically coherent?
NO

example

example (HHU09)

There exists an open set $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^3)$

example

example (HHU09)

There exists an open set $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^3)$ such that all $f \in \mathcal{U}$

example

example (HHU09)

There exists an open set $\mathcal{U} \subset \text{Diff}^1(\mathbb{T}^3)$ such that all $f \in \mathcal{U}$

- are partially hyperbolic

example

example (HHU09)

There exists an open set $\mathcal{U} \subset \mathrm{Diff}^1(\mathbb{T}^3)$
such that all $f \in \mathcal{U}$

- are partially hyperbolic
- are not dynamically coherent

theorem

theorem (HHU10)

If $f : M^3 \to M^3$ is partially hyperbolic, then

theorem

theorem (HHU10)

If $f : M^3 \to M^3$ is partially hyperbolic, then

- any invariant $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$

theorem

theorem (HHU10)

If $f : M^3 \to M^3$ is partially hyperbolic, then

- any invariant $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$
- cannot have compact leaves

non-dynamically coherent conjecture

non-dynamically coherent conjecture (HHU)

- $f : M^3 \to M^3$ non-dynamically coherent partially hyperbolic,

non-dynamically coherent conjecture

non-dynamically coherent conjecture (HHU)

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- then M is either

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harder question

f non-dynamically coherent $\overset{?}{\Rightarrow}$

non-dynamically coherent conjecture

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  1. $\mathbb{T}^3$
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harder question

f non-dynamically coherent $\overset{?}{\Rightarrow}$

- exists a compact leaf tangent to $E^s \oplus E^c$ (repelling), or

non-dynamically coherent conjecture

non-dynamically coherent conjecture (HHU)

- $f : M^3 \to M^3$ non-dynamically coherent partially hyperbolic,
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harder question

f non-dynamically coherent $\overset{?}{\Rightarrow}$

- exists a compact leaf tangent to $E^s \oplus E^c$ (repelling), or
- exists a compact leaf tangent to $E^c \oplus E^u$ (attracting).

remark

remark

- T is compact leaf tangent to $E^c \oplus E^u$

remark

remark

- T is compact leaf tangent to $E^c \oplus E^u$
- $\Rightarrow$ T Anosov torus

if this were true

if this were true

f partially hyperbolic

if this were true

if this were true

f partially hyperbolic

- in most manifolds f would be dynamically coherent

if this were true

if this were true

f partially hyperbolic

- in most manifolds f would be dynamically coherent
- in all manifolds f conservative $\Rightarrow$ dynamically coherent

if this were true

if this were true

f partially hyperbolic

- in most manifolds *f* would be dynamically coherent
- in all manifolds *f* conservative $\Rightarrow$ dynamically coherent
- (simpler conjecture)

conjectures

conjectures

non-ergodic conjecture

$f : M \to M$ non-ergodic partially hyperbolic

conjectures

non-ergodic conjecture

$f : M \to M$ non-ergodic partially hyperbolic

non-dyn. coh. conjecture

$f : M \to M$ non-dyn. coherent partially hyperbolic

conjectures

non-ergodic conjecture

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conjectures

non-ergodic conjecture

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conjectures

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?

$\exists$ Anosov torus tangent to $E^s \oplus E^u$?

conjectures

non-ergodic conjecture

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?

$\exists$ Anosov torus tangent to $E^s \oplus E^u$?

?

$\exists$ Anosov torus tangent to $E^c \oplus E^u$ or $E^c \oplus E^c$?

theorem

theorem (HHU11)

If an irreducible M contains an Anosov torus,

theorem

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reduced theorem

reduced theorem

- N^3 irreducible manifold with boundary

reduced theorem

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- N^3 irreducible manifold with boundary
- ∂N consists of Anosov tori

reduced theorem

reduced theorem

- N^3 irreducible manifold with boundary
- ∂N consists of Anosov tori
- $\Rightarrow$

$$N = \mathbb{T}^2 \times [0, 1]$$

reduced theorem $\Rightarrow$ theorem

let us prove

reduced theorem $\Rightarrow$ theorem

reduced theorem $\Rightarrow$ theorem

theorem (HHU08)

- T Anosov torus

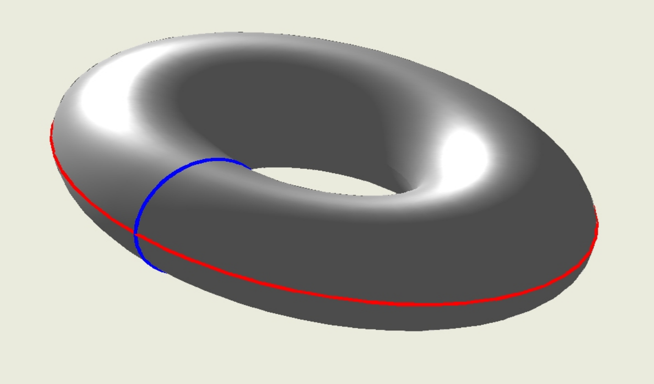
reduced theorem ⇒ theorem

theorem (HHU08)

- T Anosov torus
- $\Rightarrow T$ incompressible

reduced theorem ⇒ theorem

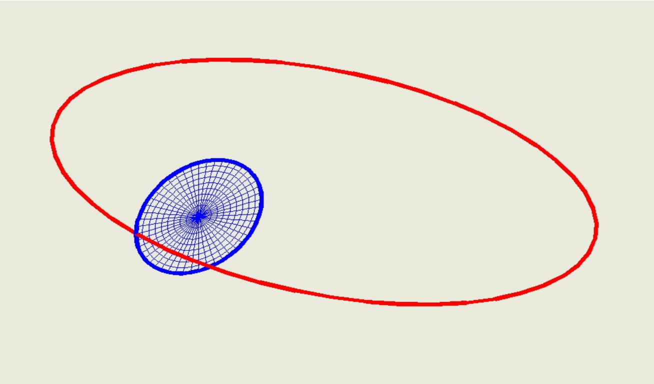
incompressible



reduced theorem $\Rightarrow$ theorem

incompressible

NO:



reduced theorem ⇒ theorem

incompressible

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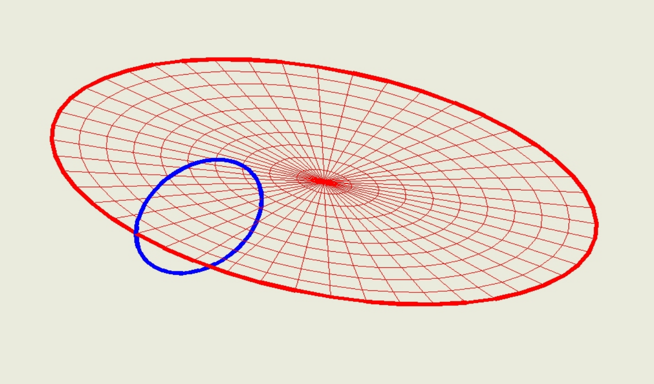

reduced theorem $\Rightarrow$ theorem

- suppose M closed irreducible

reduced theorem $\Rightarrow$ theorem

- suppose M closed irreducible
- $T \subset M$ Anosov torus

reduced theorem $\Rightarrow$ theorem

- suppose M closed irreducible
- $T \subset M$ Anosov torus
- cut M along T

reduced theorem ⇒ theorem

- suppose M closed irreducible
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- cut M along T
- $\Rightarrow$ new manifold N irreducible

reduced theorem $\Rightarrow$ theorem

- suppose M closed irreducible
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- $\partial N = T \sqcup T$ or $\partial N = T$

reduced theorem $\Rightarrow$ theorem

- suppose M closed irreducible
- $T \subset M$ Anosov torus

- cut M along T
- $\Rightarrow$ new manifold N irreducible
- $\partial N = T \sqcup T$ or $\partial N = T$
- boundary are Anosov tori

reduced theorem $\Rightarrow$ theorem

- we are in the hypotheses of the reduced theorem

reduced theorem $\Rightarrow$ theorem

- we are in the hypotheses of the reduced theorem
- $\Rightarrow N = T \times [0, 1]$

reduced theorem ⇒ theorem

- we are in the hypotheses of the reduced theorem
- $\Rightarrow N = T \times [0, 1]$
- M is a mapping torus manifold

reduced theorem $\Rightarrow$ theorem

- $\psi : T \to T$ gluing diffeomorphism such that

reduced theorem $\Rightarrow$ theorem

- $\psi : T \to T$ gluing diffeomorphism such that
- $N/\psi = M$ identifying $x \sim \psi(x)$

reduced theorem $\Rightarrow$ theorem

- $\psi : T \to T$ gluing diffeomorphism such that
- $N/\psi = M$ identifying $x \sim \psi(x)$
- $\Rightarrow \psi \circ A = A \circ \psi$

reduced theorem ⇒ theorem

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- $\Rightarrow$

reduced theorem $\Rightarrow$ theorem

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- $\Rightarrow$
  1. ψ isotopic to *id*

reduced theorem ⇒ theorem

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- $N/\psi = M$ identifying $x \sim \psi(x)$
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- $\Rightarrow$
  1. ψ isotopic to $id \to \mathbb{T}^3$

reduced theorem $\Rightarrow$ theorem

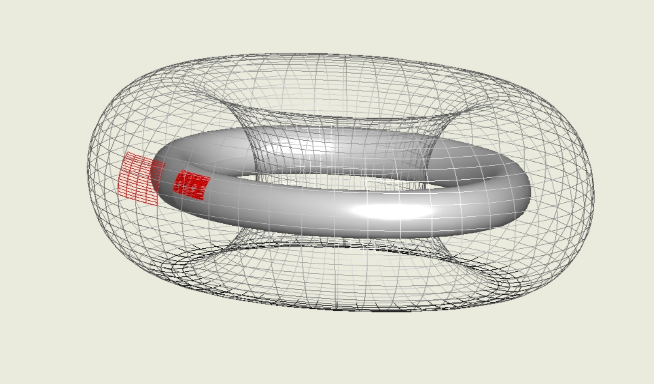
- $\psi : T \to T$ gluing diffeomorphism such that
- $N/\psi = M$ identifying $x \sim \psi(x)$
- $\Rightarrow \psi \circ A = A \circ \psi$
- $\Rightarrow$
  1. ψ isotopic to $id \to \mathbb{T}^3$
  2. ψ isotopic to $-id$

reduced theorem $\Rightarrow$ theorem

- $\psi : T \to T$ gluing diffeomorphism such that
- $N/\psi = M$ identifying $x \sim \psi(x)$
- $\Rightarrow \psi \circ A = A \circ \psi$
- $\Rightarrow$
  1. ψ isotopic to $id \to \mathbb{T}^3$
  2. ψ isotopic to $-id$
  3. ψ isotopic to hyperbolic automorphism on T

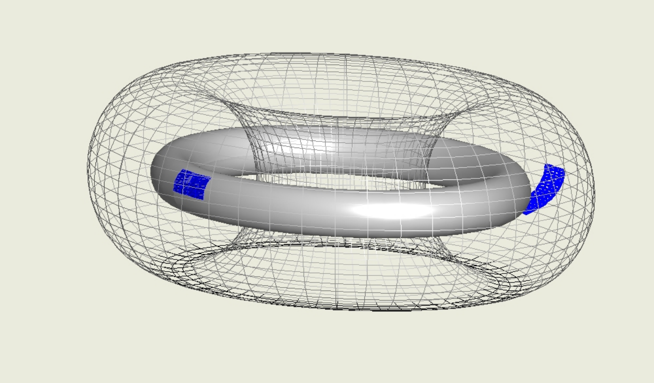
reduced theorem ⇒ theorem

① $\mathbb{T}^3$



reduced theorem ⇒ theorem

2 mapping torus of $-id$



reduced theorem ⇒ theorem

③ mapping torus of hyperbolic automorphism