Partially hyperbolic dynamics in dimension 3

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Global Dynamics beyond Uniform Hyperbolicity
Luminy, 2011

structure of the talk

theorem (HHU)

if M^3 irreducible contains an Anosov torus then M is either:

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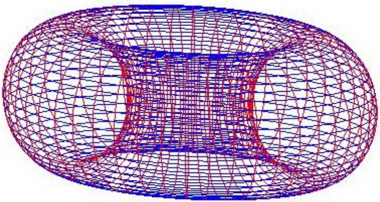
example (HHU)

there exists a non-dynamically coherent partially hyperbolic $f : \mathbb{T}^3 \to \mathbb{T}^3$

Anosov torus

Anosov torus

- T embedded 2-torus



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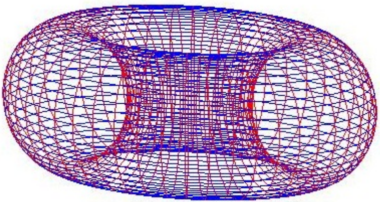

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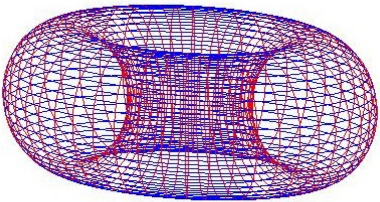

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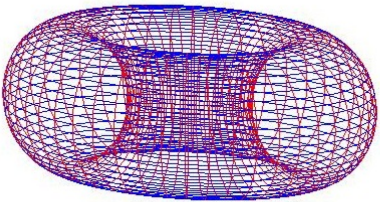

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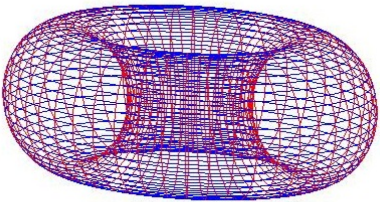

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reduced theorem

reduced theorem

- N^3 irreducible manifold with boundary

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- ∂N consists of Anosov tori

reduced theorem

reduced theorem

- N^3 irreducible manifold with boundary
- ∂N consists of Anosov tori
- $\Rightarrow$

$$N = \mathbb{T}^2 \times [0,1]$$

$\partial N \neq T$

proposition

- N orientable manifold

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proof proposition

will use

classic lemma, Lemma 3.5 Hatcher

- N orientable with boundary ∂N

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rank

rank="dimension"

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- $(Z \in \ker(i) \Rightarrow Z = \partial S \text{ in } N)$

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- $g : N \to N$ any diffeomorphism

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JSJ-decomposition

Theorem (JSJ-decomposition)

- *N^3 orientable irreducible*

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uniqueness

any minimal family $T_1, \ldots, T_n$ is unique up to isotopy

Seifert manifold

Seifert manifold

manifold supporting a foliation by circles

atoroidal manifold with boundary

atoroidal manifold

any incompressible torus is isotopic to a component of the boundary

acylindrical manifold with boundary

acylindrical manifold

any incompressible annulus with $\partial A \subset \partial N$ is isotopic to a subset of ∂N fixing ∂A

Anosov tori are incompressible

theorem (HHU)

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Anosov tori are incompressible

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- $\Rightarrow T$ incompressible

structure of our proof

- N in the hypothesis of the reduced theorem

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  1. T bounds a Seifert manifold

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- N in the hypothesis of the reduced theorem
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- either:
  1. T bounds a Seifert manifold
  2. T bounds an atoroidal and acylindrical manifold

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- N_S Seifert manifold

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- $\partial N_S \supset T$ Anosov torus

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- $\Rightarrow$

$$N_S = T \times [0, 1]$$

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remark

the previous proposition proves the reduced theorem in the case T bounds a Seifert manifold

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- $\Rightarrow T_1 = T \times \{1\}$ is redundant

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- $\Rightarrow N_S = N$

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- that are non-isotopic on ∂N_S

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classic lemma 2, Lemma 1.15 Hatcher

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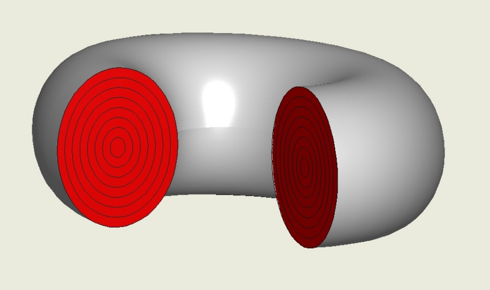
  3. $\mathbb{T}^2 \times [0,1]$ the torus cross the interval

T bounds a Seifert manifold

1 N_S = solid torus



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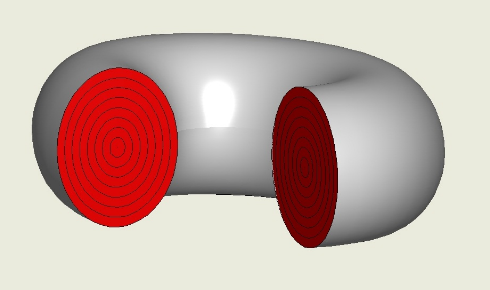

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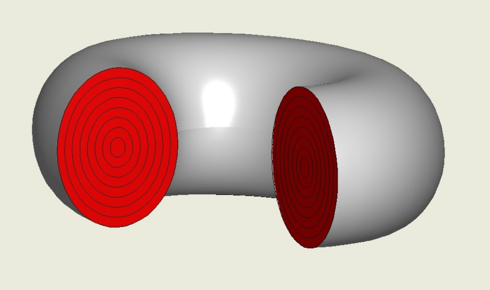

$\partial N_S = T \Rightarrow$ not possible by proposition

T bounds a Seifert manifold

② N_S = twisted I-bundle over K

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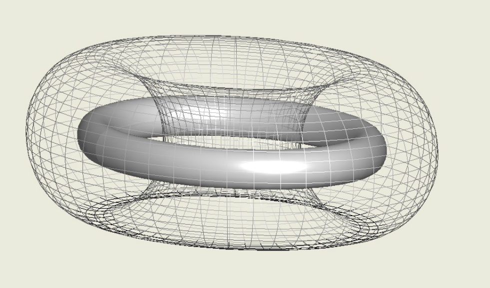
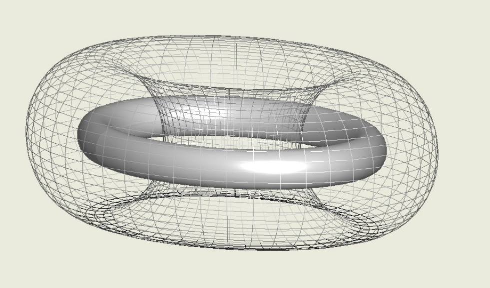
T bounds a Seifert manifold

③ $T \times [0, 1]$



T bounds a Seifert manifold

(3) $T \times [0, 1]$



the only possibility left

example

theorem

theorem (HHU10)

If $f : M^3 \to M^3$ is partially hyperbolic, then

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If $f : M^3 \to M^3$ is partially hyperbolic, then

- any invariant $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$

theorem

theorem (HHU10)

If $f : M^3 \to M^3$ is partially hyperbolic, then

- any invariant $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$
- cannot have compact leaves

structure of the example

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let us build an example $f : \mathbb{T}^3 \to \mathbb{T}^3$ such that:

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- $E^c \oplus E^u$ is uniquely integrable in $\mathbb{T}^3 \setminus T^{cu}$

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let us build an example $f : \mathbb{T}^3 \to \mathbb{T}^3$ such that:

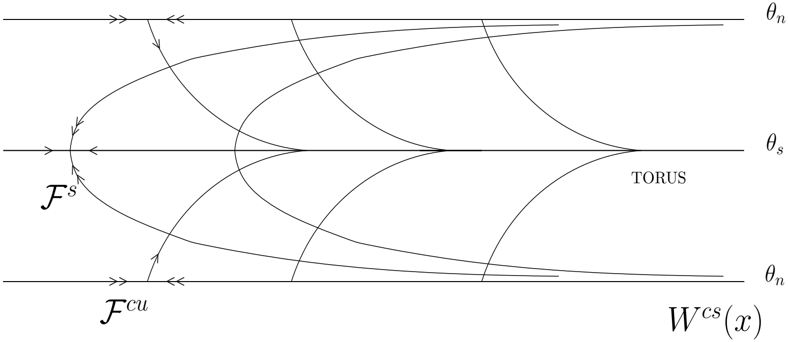
- there is an invariant torus T^{cu} tangent to $E^c \oplus E^u$
- $E^c \oplus E^u$ is uniquely integrable in $\mathbb{T}^3 \setminus T^{cu}$
- $E^c \oplus E^u$ is NOT integrable at T^{cu}

non-integrability at T^{cu}

view of a center-stable leaf

non-integrability at T^{cu}

view of a center-stable leaf



how it is built

f : Anosov $\times$ NP-SP

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$$f(x,\theta) = (Ax, \psi(\theta))$$

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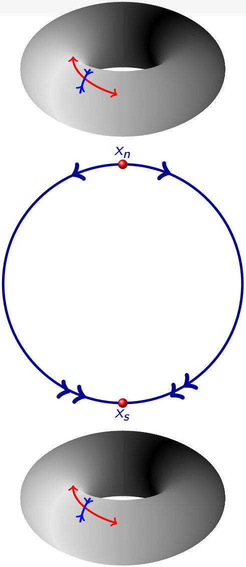

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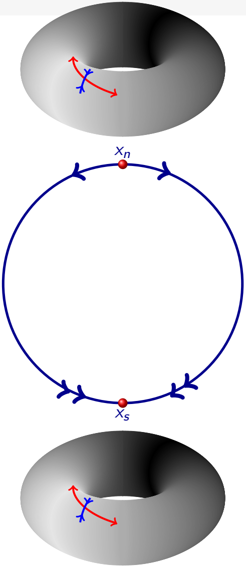
$g \sim$ Anosov $\times$ NP-SP



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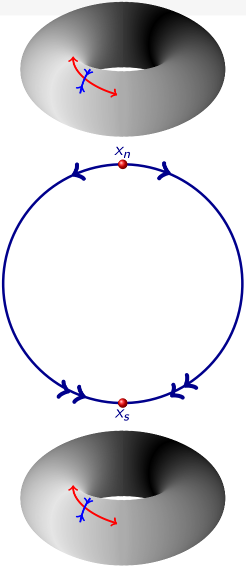
$$g(x,\theta) = (Ax + v(\theta)e_s, \psi(\theta))$$



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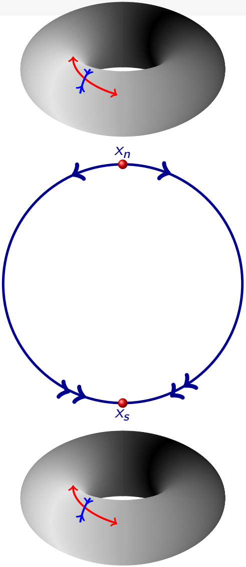
$$g(x,\theta) = (Ax + v(\theta)e_s, \psi(\theta))$$

$$v(\theta_s) = 0$$



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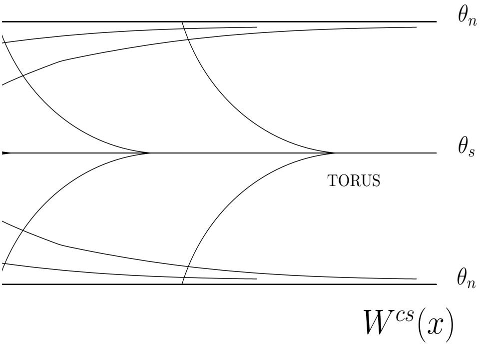

derivative of the perturbation

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$$Df(x,\theta) = \left(\begin{array}{cc|c} \lambda & 0 & 0 \\ 0 & 1/\lambda & 0 \\ \hline 0 & 0 & \psi'(\theta) \end{array}\right)$$

derivative of the perturbation

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goals

- goal 1: $g(x,\theta) = (Ax + v(\theta)e_s, \psi(\theta)) \to v$

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- goal 2: h preserves $\mathcal{F}_f^{cs}$

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- semiconjugacy: $h : \mathbb{T}^3 \to \mathbb{T}^2$
- goal 2: h preserves $\mathcal{F}_f^{cs}$
- $h(x, \theta) = x - u(\theta)e_s$

semiconjugacy

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calculations

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