Partially hyperbolic dynamics in dimension 3

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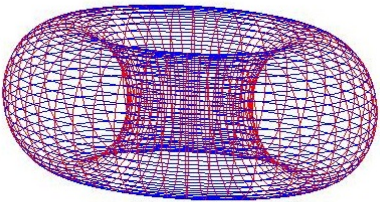
Global Dynamics beyond Uniform Hyperbolicity
Luminy, 2011

Anosov torus

Anosov torus

- T embedded 2-torus



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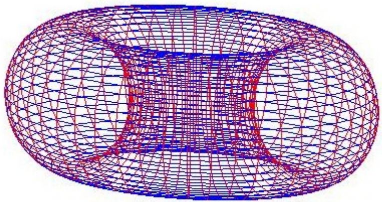

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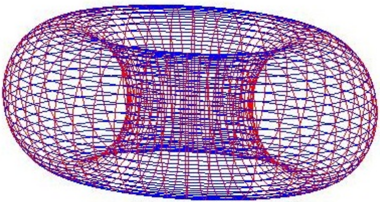

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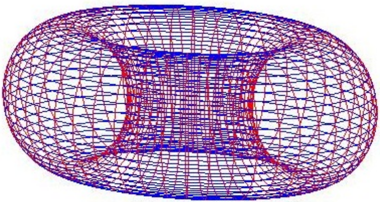

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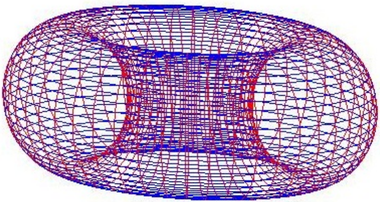
  1. $f(T) = T$
  2. $f|_T$ isotopic to Anosov



theorem

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If an irreducible M contains an Anosov torus,

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If an irreducible M contains an Anosov torus, then M is either

1. $\mathbb{T}^3$
2. the mapping torus of $-id : \mathbb{T}^2 \to \mathbb{T}^2$
3. a mapping torus of a hyperbolic automorphism of $\mathbb{T}^2$

reduced theorem

reduced theorem

- N^3 irreducible manifold with boundary

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- N^3 irreducible manifold with boundary
- ∂N consists of Anosov tori

reduced theorem

reduced theorem

- N^3 irreducible manifold with boundary
- ∂N consists of Anosov tori
- $\Rightarrow$

$$N = \mathbb{T}^2 \times [0,1]$$

JSJ-decomposition

theorem (JSJ-decomposition)

- N^3 orientable irreducible

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  1. Seifert manifold
  2. atoroidal and acylindrical

Seifert manifold

Seifert manifold

manifold supporting a foliation by circles

atoroidal manifold with boundary

atoroidal manifold

any incompressible torus is isotopic to a component of the boundary

acylindrical manifold with boundary

acylindrical manifold

any incompressible annulus with $\partial A \subset \partial N$ is isotopic to a subset of ∂N fixing ∂A

structure of the proof

- N in the hypothesis of the reduced theorem

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- $\Rightarrow \partial N$ consists of Anosov (incompressible) tori T

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proposition

proposition

An Anosov torus T cannot bound an acylindrical manifold.

strategy

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- use the global diffeo $f : M \to M$

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- that is hyperbolic on T

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- that is hyperbolic on T
- to construct an annulus A

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strategy

- use the global diffeo $f : M \to M$
- that is hyperbolic on T
- to construct an annulus A
- that is not isotopic to the boundary

step 1

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- N_A acylindrical manifold

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- $\partial N_A \supset T$ Anosov torus

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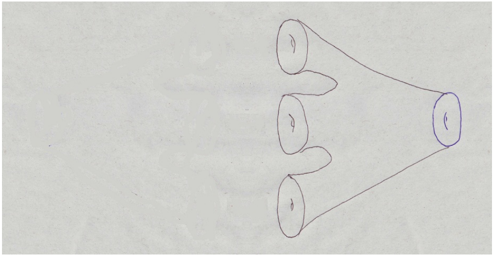

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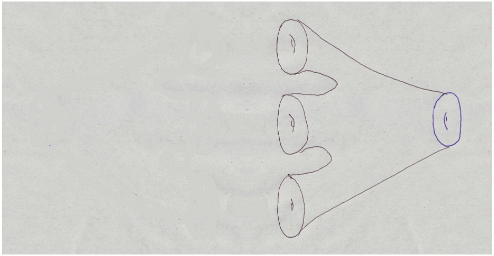
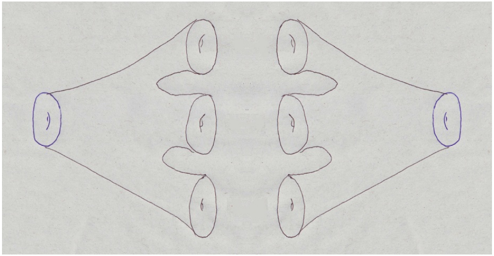


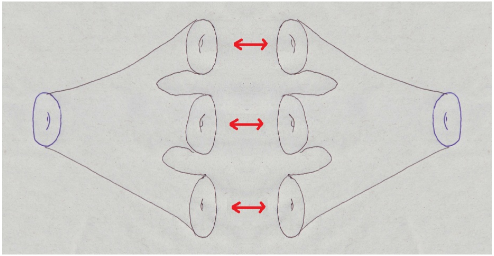
N_A

step 1



$N_A \sqcup N_A$

step 1



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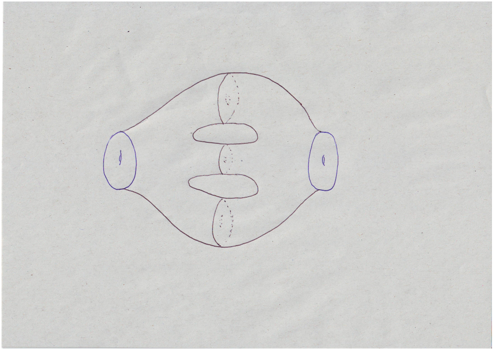

$$\partial(N_A \sqcup N_A) = T \sqcup T$$

lemma

lemma (classic)

- N orientable with boundary ∂N

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- $$\boxed{\operatorname{rank}(\ker(i)) = \frac{1}{2}\operatorname{rank}(H_1(\partial N))}$$

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- N orientable with boundary ∂N
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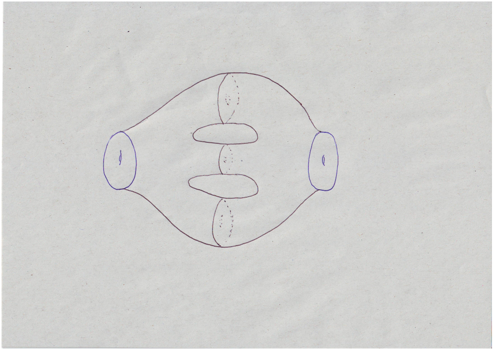
rank

rank="dimension"

step 1



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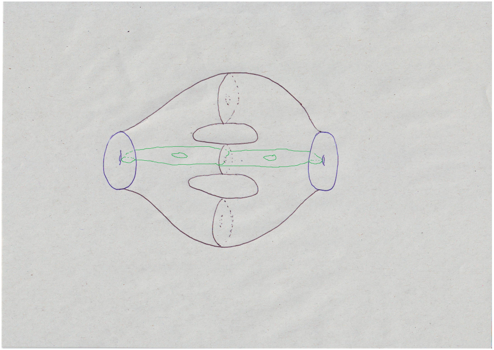

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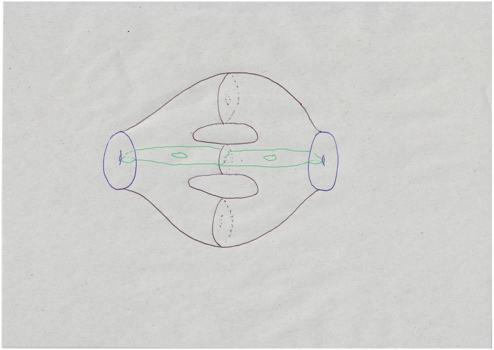
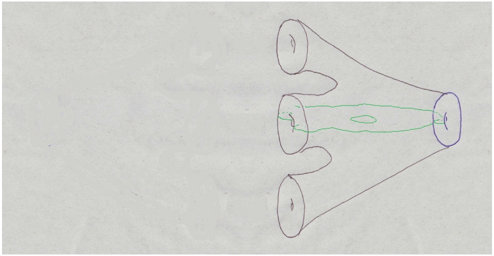


$$\partial(N_A \sqcup N_A) = T \sqcup T$$

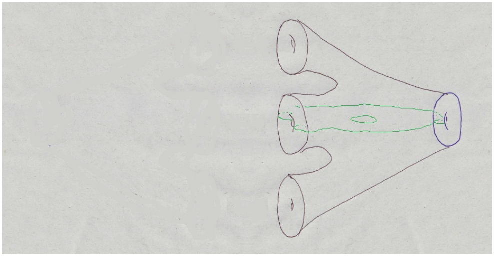
$$\partial S \subset \ker(i)$$

step 1



$\partial N_A \supset T$

step 1



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- exists S surface such that
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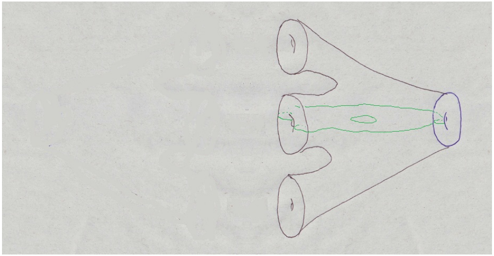

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- exists S surface such that
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  2. S is not an annulus isotopic to the boundary

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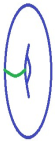

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2. $\partial_3 R \subset \partial S \subset \partial N_A$

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4. $\partial_2 R$ and $\partial_4 B$ isotopic in $f^N(S)$

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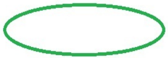
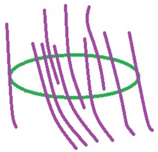
- $k = \max \#\{\text{disjoint non-isotopic curves in } S\}$



$\gamma \subset S \cap T$

step 2

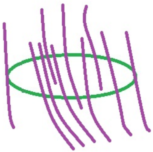
- $\gamma \subset S \cap T$ essential
- $k = \max \#\{\text{disjoint non-isotopic curves in } S\}$
- $\#(\gamma \cap f^N(\gamma)) \geq 2(k+1)^2$



$\gamma \cap f^N(\gamma)$

step 2

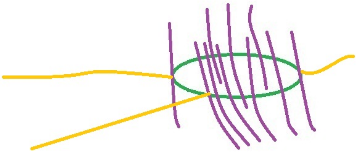
- we may assume $f^N(S) \pitchfork S$



$\gamma \cap f^N(\gamma)$

step 2

- we may assume $f^N(S) \pitchfork S$
- consider curves in $S \cap f^N(S)$



$S \cap f^N(S)$

step 2

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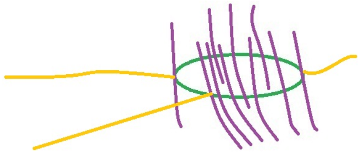
- there are $\geq (k+1)^2$ disjoint curves with endpoints in $\gamma \cap f^N(\gamma)$



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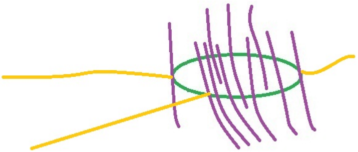

$S \cap f^N(S)$

step 2

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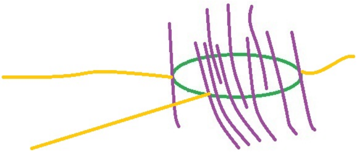
- there are $\geq (k+1)^2$ disjoint curves with endpoints in $\gamma \cap f^N(\gamma)$
- there are $\geq (k+1)$ disjoint such curves isotopic in S
- there are at least 2 such curves isotopic in S and $f^N(S)$



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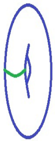

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the annulus

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the annulus

- $A = R \cup R'$ where

the annulus

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- $A = R \cup R'$ where
- R obtained in step 2

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- $A = R \cup R'$ where
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- R' subset in $f^N(S)$ such that
  1. $\partial_1 R' \subset f^N(\gamma) \subset T$

the annulus

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- $A = R \cup R'$ where
- R obtained in step 2
- R' subset in $f^N(S)$ such that
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  2. $\partial_3 R' \subset \partial(f^N(S))$

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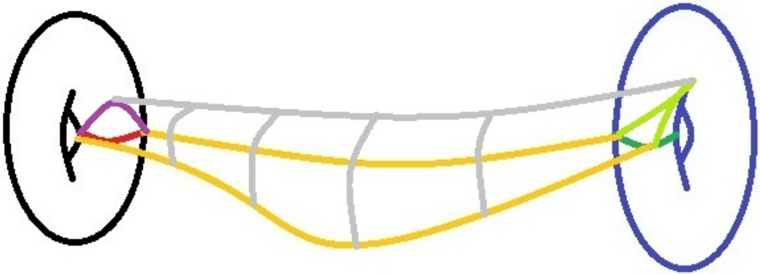

step 3

step 3

A is not isotopic to the boundary

remark

remark

each loop $\subset A \cap T$ is homotopically non-trivial

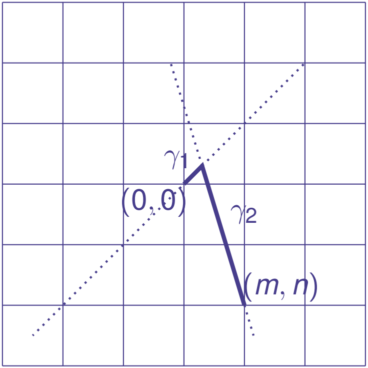

Figure: a closed curve in T

property

property [Lemma 1.10, Hatcher]

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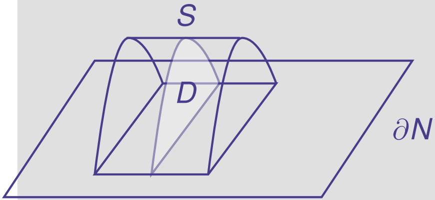

property

property [Lemma 1.10, Hatcher]

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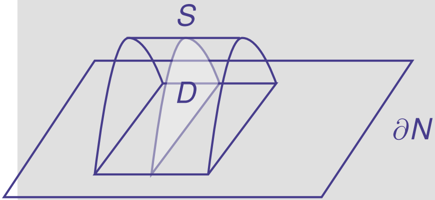

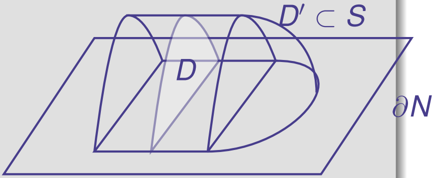

proof step 3

- suppose A isotopic to ∂N

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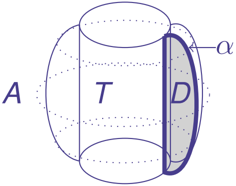
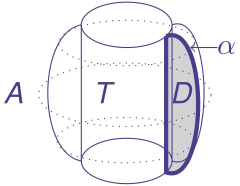


- $\alpha \subset S \subset f^N(S)$

proof step 3

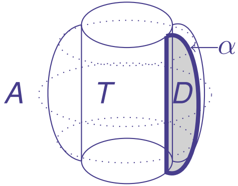
- suppose A isotopic to ∂N



- $\alpha \subset S \subset f^N(S)$
- one endpoint of α in $\gamma \cap f^N(\gamma)$

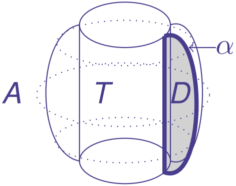
proof step 3

- suppose A isotopic to ∂N



- $\alpha \subset S \subset f^N(S)$
- one endpoint of α in $\gamma \cap f^N(\gamma)$
- the other endpoint in another connected component of ∂A

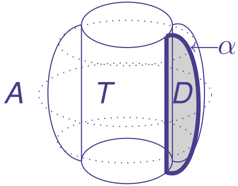
proof step 3

- suppose A isotopic to ∂N
- 

$\alpha \subset S \subset f^N(S)$
- one endpoint of α in $\gamma \cap f^N(\gamma)$
- the other endpoint in another connected component of ∂A
- by property $\exists D_1 \subset S$ and $D_2 \subset f^N(S)$

proof step 3

- suppose A isotopic to ∂N



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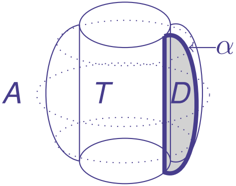
- one endpoint of α in $\gamma \cap f^N(\gamma)$
- the other endpoint in another connected component of ∂A
- by property $\exists D_1 \subset S$ and $D_2 \subset f^N(S)$
- with $E = D_1 \cup D_2$ and $\partial E \subset \gamma \cup f^N(\gamma)$

proof step 3

- suppose A isotopic to ∂N
- 

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- with $E = D_1 \cup D_2$ and $\partial E \subset \gamma \cup f^N(\gamma)$
- ABSURD

to finish

so

to finish

so

- step 1

to finish

so

- step 1
- step 2

to finish

so

- step 1
- step 2
- step 3

to finish

so

- step 1
- step 2
- step 3
- $\Rightarrow$ T cannot bound an acylindrical component

to finish

so

- step 1
- step 2
- step 3
- $\Rightarrow$ T cannot bound an acylindrical component
- $\Rightarrow$ theorem

further questions

further questions

plaque expansiveness

does dynamical coherence $\Rightarrow$ plaque expansiveness?

further questions

plaque expansiveness

does dynamical coherence $\Rightarrow$ plaque expansiveness?

?

$\nexists$ Anosov torus $\Rightarrow$ plaque expansiveness?

advances

theorem (Hammerlindl)

advances

theorem (Hammerlindl)

- f absolutely partially hyperbolic

advances

theorem (Hammerlindl)

- f absolutely partially hyperbolic
- $\mathcal{F}^s$ and $\mathcal{F}^u$ quasi-isometric

advances

theorem (Hammerlindl)

- f absolutely partially hyperbolic
- $\mathcal{F}^s$ and $\mathcal{F}^u$ quasi-isometric
- $\Rightarrow$ plaque expansiveness

thanks

THANKS!