Some advances on partially hyperbolic and non-uniformly hyperbolic dynamics

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setting

- M compact Riemannian manifold

setting

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- $f : M \to M$ diffeomorphism

setting

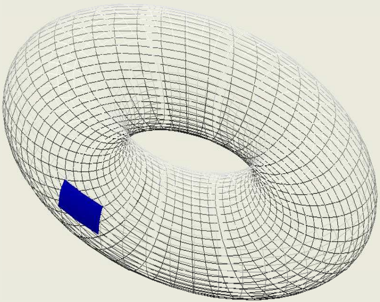
- M compact Riemannian manifold
- $f : M \to M$ diffeomorphism
- f preserves smooth volume (conservative)

problem

How frequent is ergodicity?

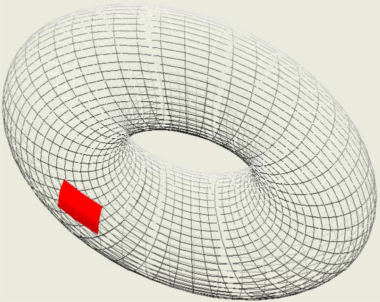
fragility of ergodicity

Irrational rotation on the torus



ergodic

Rational rotation on the torus



non ergodic

stable ergodicity

Definition

$f : M \to M$ is stably ergodic if

stable ergodicity

Definition

$f : M \to M$ is stably ergodic if

$$g \sim f \qquad \Rightarrow \qquad g \quad \text{ergodic}$$

problem

$$(*) \quad \stackrel{?}{\Longrightarrow} \quad \text{stable ergodicity}$$

hyperbolicity

Definition

$f : M \to M$ is hyperbolic if

hyperbolicity

Definition

$f : M \to M$ is hyperbolic if

$$TM = E^s \oplus E^u$$

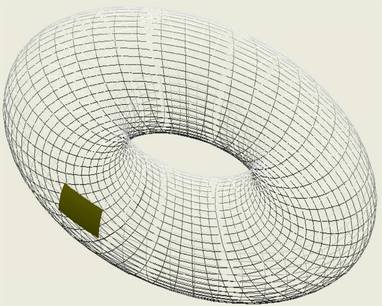
hyperbolicity

Definition

$f : M \to M$ is hyperbolic if

$$\begin{array}{ccccc} TM & = & E^s & \oplus & E^u \\ & & \uparrow & & \uparrow \\ & & Tf \text{ contracting} & & Tf \text{ expanding} \end{array}$$

hyperbolicity implies stable ergodicity



Anosov (1967)

example

Example (Grayson-Pugh-Shub (1994))

stable ergodicity $\not\Rightarrow$ hyperbolicity

example

Example (Grayson-Pugh-Shub (1994))

stable ergodicity $\not\Rightarrow$ hyperbolicity

- partially hyperbolic

partial hyperbolicity

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$$TM = E^s \oplus E^c \oplus E^u$$

partial hyperbolicity

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$f : M \to M$ is partially hyperbolic if

$$TM = E^s \oplus E^c \oplus E^u$$

E^s ↑ Tf contracting

E^u ↑ Tf expanding

conjecture

Conjecture Pugh-Shub (1995)

partial hyperbolicity “ $\Rightarrow$ " stable ergodicity

conjecture

Conjecture: Pugh-Shub (1995)

Stable ergodicity is C^r-dense in $\mathcal{P}H_m(M)$

Pugh-Shub program

Conjecture A

Accessibility “⇒" ergodicity

Pugh-Shub program

Conjecture A

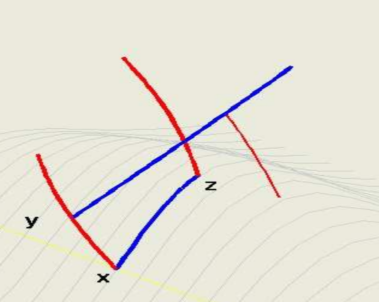
Accessibility “$\Rightarrow$" ergodicity

Conjecture B

Stable accessibility is C^r-dense in $\mathcal{PH}_m(M)$

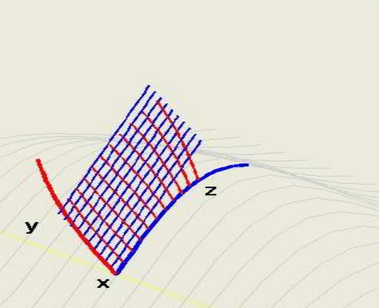
accessibility class

Definition (Accessibility class)



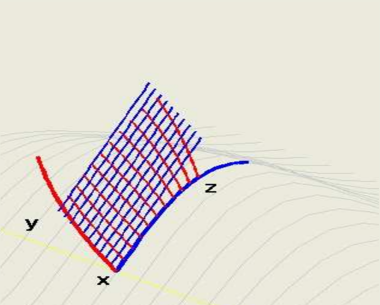
accessibility

Example (non-accessible)

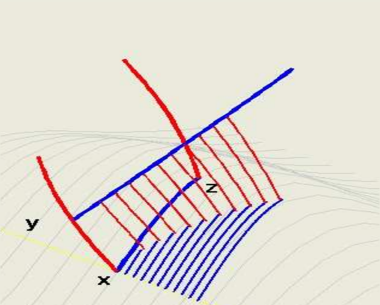


accessibility

Example (non-accessible)



Example (accessible)



advances

Theorem (Burns-Dolgopyat-Pesin (2002))

$f : M \to M$ partially hyperbolic

advances

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- *accessibility*

advances

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- *center Lyapunov exponents > 0*

advances

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$\Rightarrow$ stable ergodicity

advances

Theorem (Burns-Wilkinson (2006))

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- *accessibility*

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Theorem (Burns-Wilkinson (2006))

$f : M \to M$ partially hyperbolic

- *accessibility*
- *bunching condition*

advances

Theorem (Burns-Wilkinson (2006))

$f : M \to M$ partially hyperbolic

- *accessibility*
- *bunching condition*

$\Rightarrow$ *ergodicity*

advances

Theorem (Dolgopyat-Wilkinson (2003))

Stable accessibility is C^1-dense among $\mathcal{P}H_*(M)$

advances

Theorem (FRH-JRH-Ures, Burns-FRH-JRH-Talitskaya-U (2008))

- $\dim E^c = 1$

advances

Theorem (FRH-JRH-Ures, Burns-FRH-JRH-Talitskaya-U (2008))

- dim $E^c = 1$

Stable accessibility is C^∞ dense in $\mathcal{P}H_*(M)$

advances

Theorem (F. Rodriguez Hertz-JRH-Ures (2008))

- $\dim E^c = 1$

advances

Theorem (F. Rodriguez Hertz-JRH-Ures (2008))

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Stable ergodicity is C^∞ dense in $\mathcal{P}H_m(M)$

advances

Theorem (FRH-JRH-Tahzibi-U (2009))

- $\dim E^c = 2$

advances

Theorem (FRH-JRH-Tahzibi-U (2009))

- $\dim E^c = 2$

Stable ergodicity is C^1-dense in $\mathcal{P}H_m(M)$

remark

Example (Tahzibi)

stable ergodicity $\not\Rightarrow$ partial hyperbolicity

remark

Proposition (Arbieto-Matheus)

stable ergodicity $\Rightarrow$ dominated splitting

domination

Definition

$f : M \to M$ has a dominated splitting if

domination

Definition

$f : M \to M$ has a dominated splitting if

$$TM = E \oplus F$$

domination

Definition

$f : M \to M$ has a dominated splitting if

$$TM = \underset{\substack{\uparrow \\ Tf \text{ more contracting}}}{E} \oplus \underset{\substack{\uparrow \\ Tf \text{ more expanding}}}{F}$$

questions

questions

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- is accessibility condition open when $c > 1$? (Didier: $c = 1$)

questions

questions

- is accessibility condition open when $c > 1$? (Didier: $c = 1$)
- are accessibility classes manifolds? (RHRHU $c = 1$)

questions

question

questions

question

- stable ergodicity C^1-dense in $\mathcal{P}H_m(M)$ with $c > 2$?

questions

question

domination + (?) $\Rightarrow$ ergodicity?

problem

C^1 generic ergodicity?

conjecture Avila-Bochi

Conjecture [Avila-Bochi]

For generic $f \in \mathrm{Diff}^1_m(M)$, either

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  - $LE(x)|_{E^+} > 0$ and $LE(x)|_{E^-} < 0$ m-a.e.

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  - f is ergodic

advances

Theorem (Mañé-Bochi (2002))

For generic $f \in \mathrm{Diff}^1_m(M^2)$ *either*

advances

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advances

Theorem (Mañé-Bochi (2002))

For generic $f \in \mathrm{Diff}^1_m(M^2)$ *either*

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- f *is Anosov*

advances

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For generic $f \in \mathrm{Diff}^1_m(M^3)$*, either*

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advances

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advances

Theorem (JRH (2009))

For generic $f \in \mathrm{Diff}^1_m(M^3)$*, either*

- $LE(x) = 0$ *m-a.e., or else*
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    - f *is ergodic*
    - $Nuh(f) \doteq M$

advances

Theorem (Avila-Bochi (2010))

For generic $f \in \mathrm{Diff}^1_m(M^n)$*, either*

advances

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advances

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  - $f|_{Nuh(f)}$ *ergodic*

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  - $f|_{Nuh(f)}$ *ergodic*
  - *m-a.e. $x \in Nuh(f)$, $o(x)$ is dense*

question

question

question

question

- is stable ergodicity C^1-dense among $\mathcal{N}uh(M)$?

question

question

- is stable ergodicity C^1-dense among $\mathcal{N}uh(M)$?
- if $\dim M = 3$?

question

question

question

question

- $\exists$ C^1-dense set of smooth diffeos in $\mathcal{N}uh(M)$?

question

question

- $\exists$ C^1-dense set of smooth diffeos in $\mathcal{N}uh(M)$?
- $\dim M = 3$?

question

question

question

question

- $LE \overset{\circ}{=} 0$ vs. ergodicity in the symplectic setting?