Some advances on partially hyperbolic and non-uniformly hyperbolic dynamics

Jana Rodriguez Hertz

Universidad de la República
Uruguay

Dijon 2010

setting

- M^3 compact Riemannian 3-manifold

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- $f : M \to M$ partially hyperbolic diffeomorphism

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- $f : M \to M$ partially hyperbolic diffeomorphism
- f preserves smooth volume (conservative)

partial hyperbolicity

Definition

$f : M^3 \to M^3$ is partially hyperbolic

$$TM = E^s \oplus E^c \oplus E^u$$

partial hyperbolicity

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$$\begin{array}{ccccccc} TM & = & E^s & \oplus & E^c & \oplus & E^u \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \text{1-dim} & & \text{1-dim} & & \text{1-dim} \end{array}$$

problem

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For which M^3 do we have

$$\{\text{partially hyperbolic}\} \subset \{\text{ergodic}\} \quad ?$$

conjecture

Conjecture (RHRHU)

If M^3 is such that

conjecture

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Then, either

- $M = \mathbb{T}^3$

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Then, either

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- M is the mapping torus of $-id$, or
- M is the mapping torus of a hyperbolic toral automophism of $\mathbb{T}^2$

nilmanifolds

- On nilmanifolds the conjecture is true.

nilmanifolds

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If M^3 is a nilmanifold such that

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then $M = \mathbb{T}^3$

hint 1

Theorem (RHRHU (2010))

$f : M^3 \to M^3$ such that

- *there exists a periodic su-torus.*

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- *there is a su-lamination, extensible to a foliation without compact leaves (*)*

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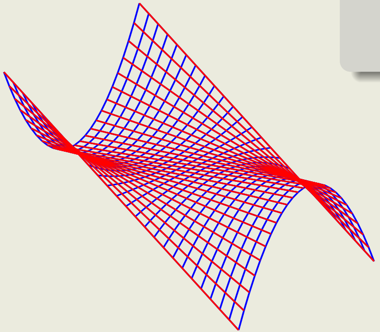
$f \in \mathcal{PH}_m(M^3)$ not accessible. Then either

- *there exists a periodic su-torus*
- *there is a su-lamination, extensible to a foliation without compact leaves (*)*
- *there is an su-foliation without Reeb components*

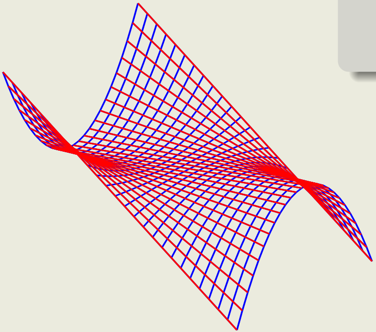
accessibility



question

$f \in \mathcal{P}H_m(f)$ non accessible

accessibility

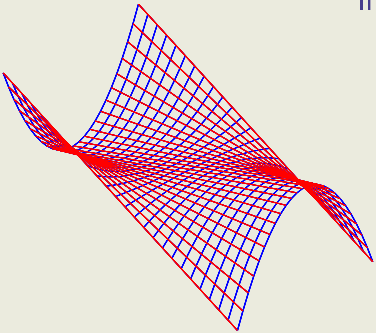


question

$f \in \mathcal{P}H_m(f)$ non accessible

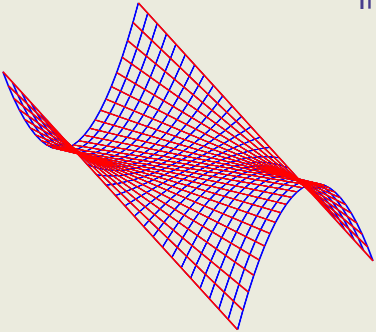
- $\exists$ *su*-lamination extensible to a foliation without compact leaves?

accessibility



If L is an *su*-leaf such that:

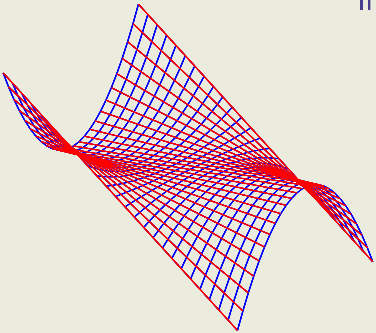
accessibility



If L is an *su*-leaf such that:

- L is complete

accessibility



If L is an *su*-leaf such that:

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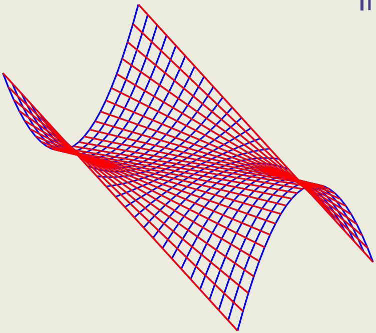

If L is an *su*-leaf such that:

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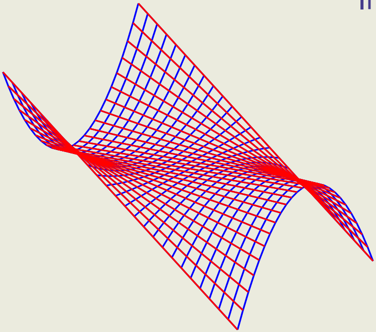

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- f Anosov on L (uniform angles)

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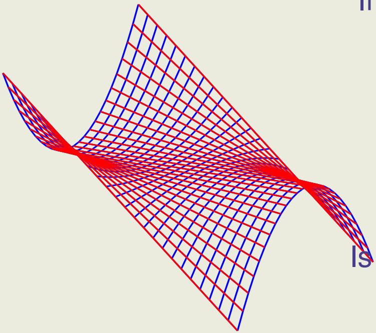

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Is $L = \mathbb{T}^2$?

remark

remark/question

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Then, either

- M is the mapping torus of a hyperbolic toral automorphism of $\mathbb{T}^2$
- the foliation is minimal (essential accessibility??)

questions

question

How is the dynamics if

$$\{\text{partially hyperbolic}\} \not\subset \{\text{ergodic}\}?$$

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- Hammerlindl-Ures (2010) advances in $\mathbb{T}^3$

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- f transitive

dynamical coherence

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Theorem (Brin(2002))

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- $W^s_R(\gamma) \cap W^u_R(\gamma) \setminus \gamma$ *contains a center circle*

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then

- *f is dynamically coherent*
- *$f \sim$ skew product over hyperbolic automorphism on $\mathbb{T}^2$*

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- Parwani (2010): generalized it to nilmanifolds

example

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$\exists\, f \in \mathcal{P}H(\mathbb{T}^3)$ such that

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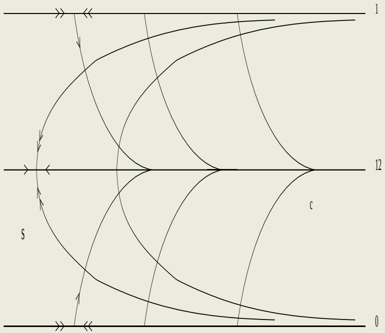
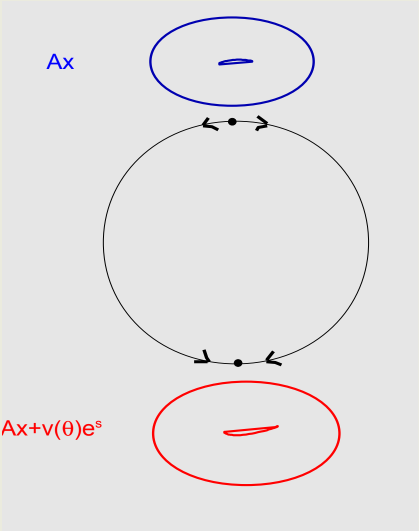


$W^{cs}(x, \theta)$

example

Example

$$F(x,\theta) = (Ax + v(\theta)e^s, f(\theta))$$



question

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can we avoid “branchings" in the transitive setting?