# Partially hyperbolic dynamics in dimension 3

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# partial hyperbolicity

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- $|Df(x)v^s| < |Df(x)v^c| < |Df(x)v^u|$

# problem

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describe the non-ergodic partially hyperbolic diffeomorphisms in dimension 3

# conjecture

## conjecture [RHRHU08]

If $f : M^3 \to M^3$ is a conservative non-ergodic partially hyperbolic diffeomorphism,

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3. a mapping torus of a hyperbolic automorphism of $\mathbb{T}^2$.

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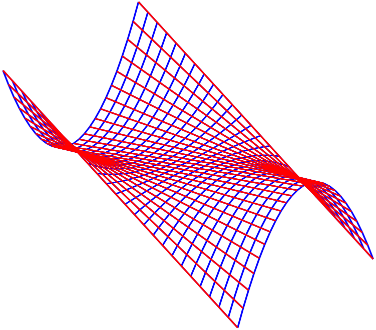

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2. $f = \text{Anosov} \times NPSP : \mathbb{T}^2 \times \mathbb{T}^1 \hookleftarrow$
   - $\Rightarrow$ all accessibility classes are tori (not invariant)

# accessibility

**Definition (accessibility)**

$f : M \to M$ has the accessibility property

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$f : M \to M$ has the accessibility property if
$M$ is an accessibility class

# theorem

**Theorem (BW10, RHRHU08)**

*$f : M^3 \to M^3$ conservative and partially hyperbolic*

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*$f : M^3 \to M^3$ conservative and partially hyperbolic*

- *accessibility $\Rightarrow$ essential accessibility $\Rightarrow$ ergodicity*

# problem

$$\{\text{non ergodic}\} \subset \{\text{non accessible}\}$$

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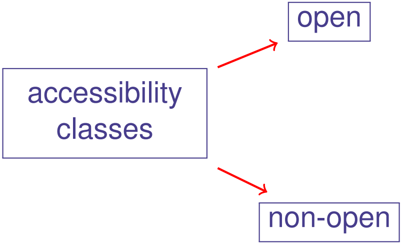

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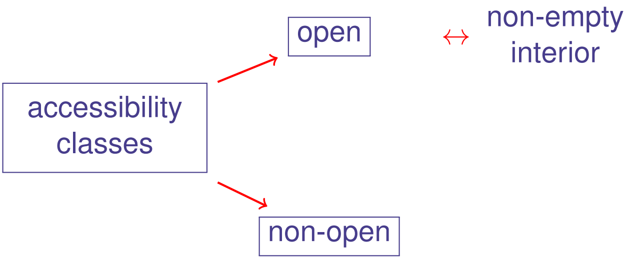

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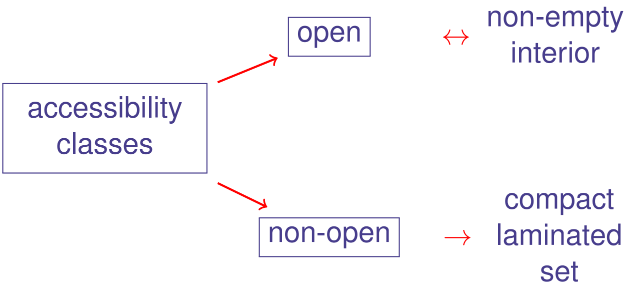

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*If* $f : M^3 \to M^3$ *conservative partially hyperbolic, and*

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3. *there exists a minimal foliation by accessibility classes*

# remark

In case ⓶, each boundary leaf of the lamination:

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- stable and unstable manifolds of periodic points are dense w.r.t. the intrinsic topology

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- If $f$ is dynamically coherent, then
- $\exists$ invariant foliation $\mathcal{F}^c$

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**remark**

- If $f$ is dynamically coherent, then
- $\exists$ invariant foliation $\mathcal{F}^c$
- that sub-foliates $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$

# question

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If $f : M^3 \to M^3$ is partially hyperbolic, is $f$ dynamically coherent?

# theorem

**Theorem (Brin03)**

*If $M \to M$ absolutely partially hyperbolic,*

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**Theorem (Brin-Burago-Ivanov08)**

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**Theorem (Brin-Burago-Ivanov08)**

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- *$\Rightarrow$ $f$ is uniquely dynamically coherent*

**Theorem (Parwani10)**

*Extended the result to 3-nilmanifolds*

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*If $f : M^3 \to M^3$ is partially hyperbolic, then*

- *any invariant $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$*
- *cannot have compact leaves*

# remarks

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2. If $T$ is invariant, then $f : T \to T$ is isotopic to Anosov

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If there exists $f : M^3 \to M^3$ (ph or not) such that:

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Is it true that if $N^3 \neq \mathbb{T}^3$, then

all $f : N^3 \to N^3$ ph are dynamically coherent?