Partially hyperbolic dynamics in dimension 3

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dynamical coherence

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  1. $\exists$ invariant foliation $\mathcal{F}^{cs}$ tangent to $E^s \oplus E^c$
  2. $\exists$ invariant foliation $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$

remark

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- If f is dynamically coherent, then
- $\exists$ invariant foliation $\mathcal{F}^c$
- that sub-foliates $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$

question

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If $f : M^3 \to M^3$ is partially hyperbolic, is f dynamically coherent?

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theorem

Theorem (RHRHU10)

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If $f : M^3 \to M^3$ is partially hyperbolic, then

- *any invariant $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$*

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If $f : M^3 \to M^3$ is partially hyperbolic, then

- *any invariant $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$*
- *cannot have compact leaves*

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- f has a compact invariant torus T^{cu} tangent to $E^c \oplus E^u$
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- $E^c \oplus E^u$ is uniquely integrable in $\mathbb{T}^3 \setminus T^{cu}$
- $E^c \oplus E^u$ is NOT integrable at T^{cu}

non-integrability at T^{cu}

view of a center-stable leaf

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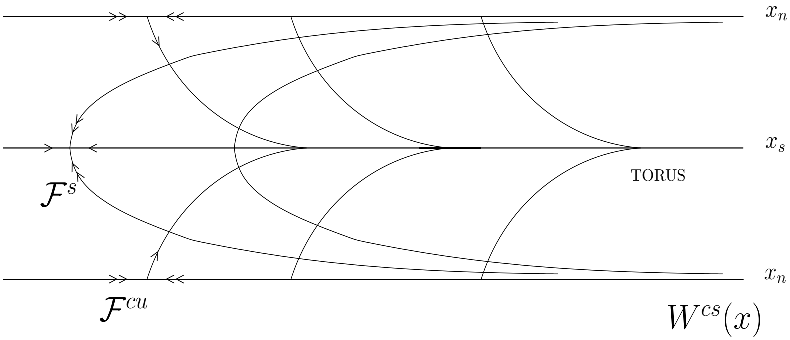

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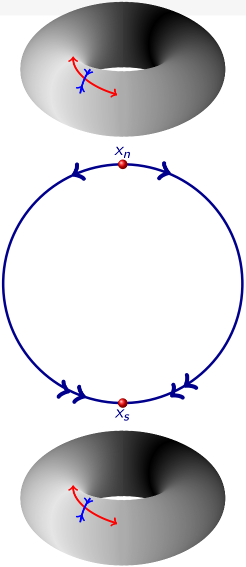

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$g \sim$ Anosov $\times$ NP-SP

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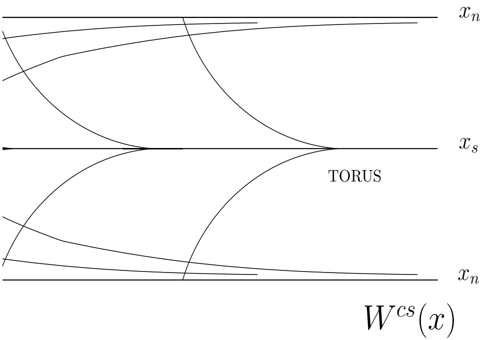

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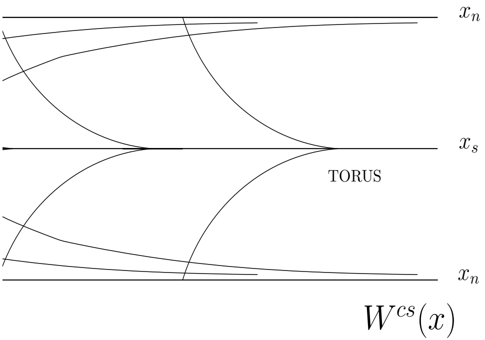

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derivative of the perturbation

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- $h(x,\theta) = x - u(\theta)e_s$

semiconjugacy

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$$h(Ax + v(\theta)e_s, \psi(\theta)) \quad = \quad Ah(x,\theta)$$

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$$\boxed{u \circ \psi = \lambda u + v}$$

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- $\Rightarrow u'$ continuous for $\theta \neq x_s$

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- curves $\theta \mapsto (x + u(\theta)e_s, \mathit{theta})$ invariant partition

question

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$f : M^3 \to M^3$ dynamically coherent is plaque expansive?

remark

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- is K invariant for ϕ?