Ergodic partially hyperbolic systems of 3-manifolds

F. Rodriguez Hertz

M.A. Rodriguez Hertz

R. Ures

Question

How frequent is ergodicity among conservative systems?

Question

How frequent is ergodicity among conservative systems?

KAM
phenomenon

Conjecture

CONJECTURE [PUGH,SHUB] *Stable ergodicity is C^r dense among conservative partially hyperbolic systems*

Pugh-Shub program

CONJECTURE A : *Accessibility* $\Rightarrow$ *ergodicity in* $\mathrm{PH}^2_m(M)$

Pugh-Shub program

CONJECTURE A : *Accessibility* $\Rightarrow$ *ergodicity in* $\mathrm{PH}^2_m(M)$

CONJECTURE B : *Accessibility is* C^r *open and dense among partially hyperbolic systems*

in 3-manifolds

THEOREM *Stable ergodicity is* C^r *dense in* $\mathrm{PH}^r_m(M^3)$

problem

Describe the non-ergodic dynamics in $\mathrm{PH}^r_m(M^3)$.

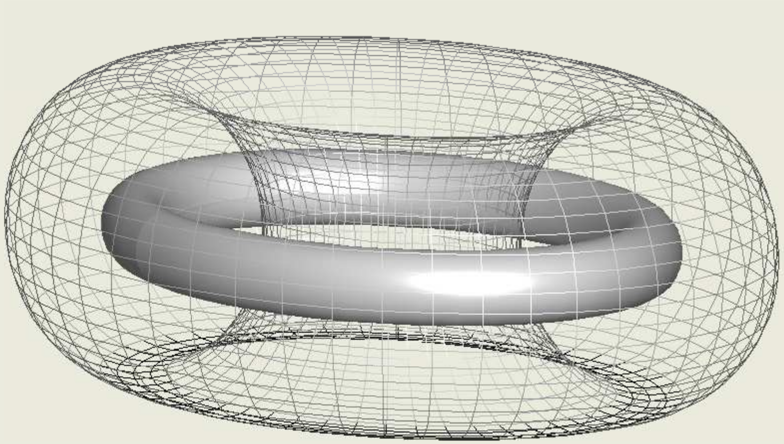
conjecture

M^3 supports a non-ergodic diffeomorphism in $\mathrm{PH}^r_m(M^3)$

conjecture

M^3 supports a non-ergodic diffeomorphism in $\mathrm{PH}^r_m(M^3)$

$$\Downarrow$$



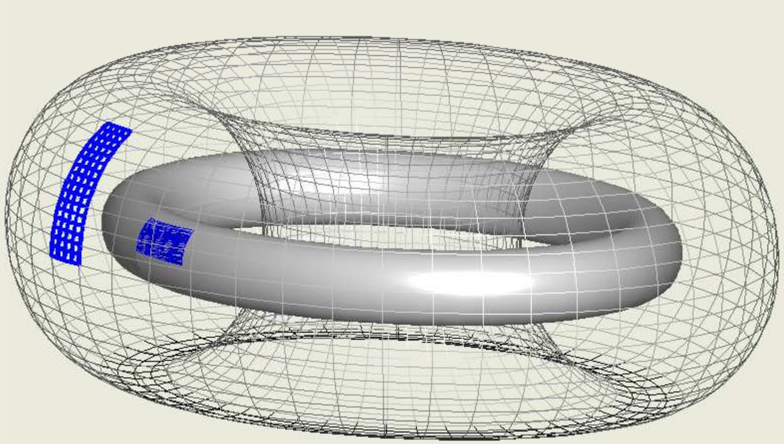
M is a mapping torus:

$$M = \mathbb{T}^2 \times [0,1]/(x,1) \sim (Ax,0)$$

conjecture

M^3 supports a non-ergodic diffeomorphism in $\mathrm{PH}^r_m(M^3)$

$$\Downarrow$$



M is a mapping torus:

$$M = \mathbb{T}^2 \times [0,1]/(x,1) \sim (Ax,0)$$

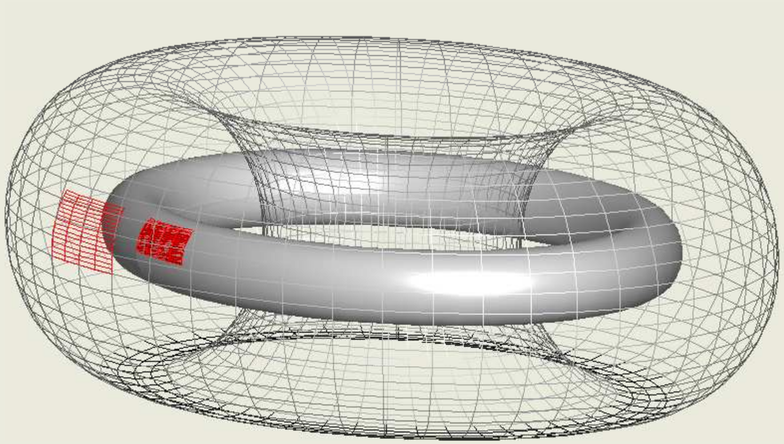
with

- $A : \mathbb{T}^2 \to \mathbb{T}^2$ hyperbolic or

conjecture

M^3 supports a non-ergodic diffeomorphism in $\mathrm{PH}^r_m(M^3)$

$$\Downarrow$$



M is a mapping torus:

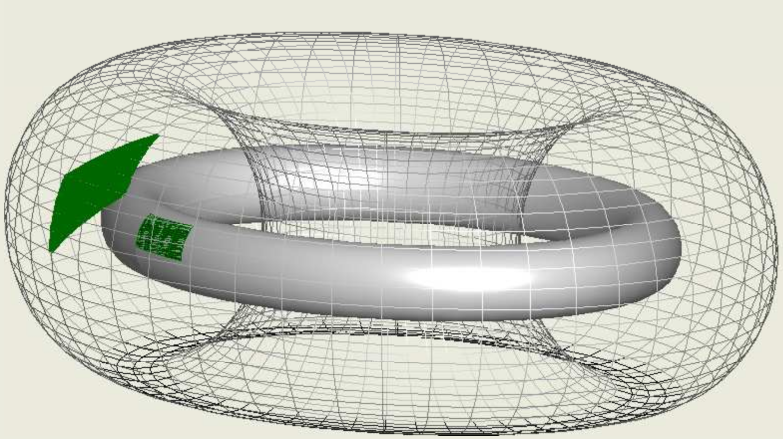
$$M = \mathbb{T}^2 \times [0,1]/(x,1) \sim (Ax,0)$$

with

- $A : \mathbb{T}^2 \to \mathbb{T}^2$ hyperbolic or
- $A = \pm$id

theorem

THEOREM A $M^3 \neq \mathbb{T}^3$ *nilmanifold* $\implies$ $\mathrm{PH}^r_m(M^3) \subset \{$ *ergodic* $\}$



M is a mapping torus:

$$M = \mathbb{T}^2 \times [0,1]/(x,1) \sim (Ax,0)$$

with

$$A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

accessibility classes

$AC(x) =$ accessibility class of x

accessibility classes

$AC(x) =$ accessibility class of x

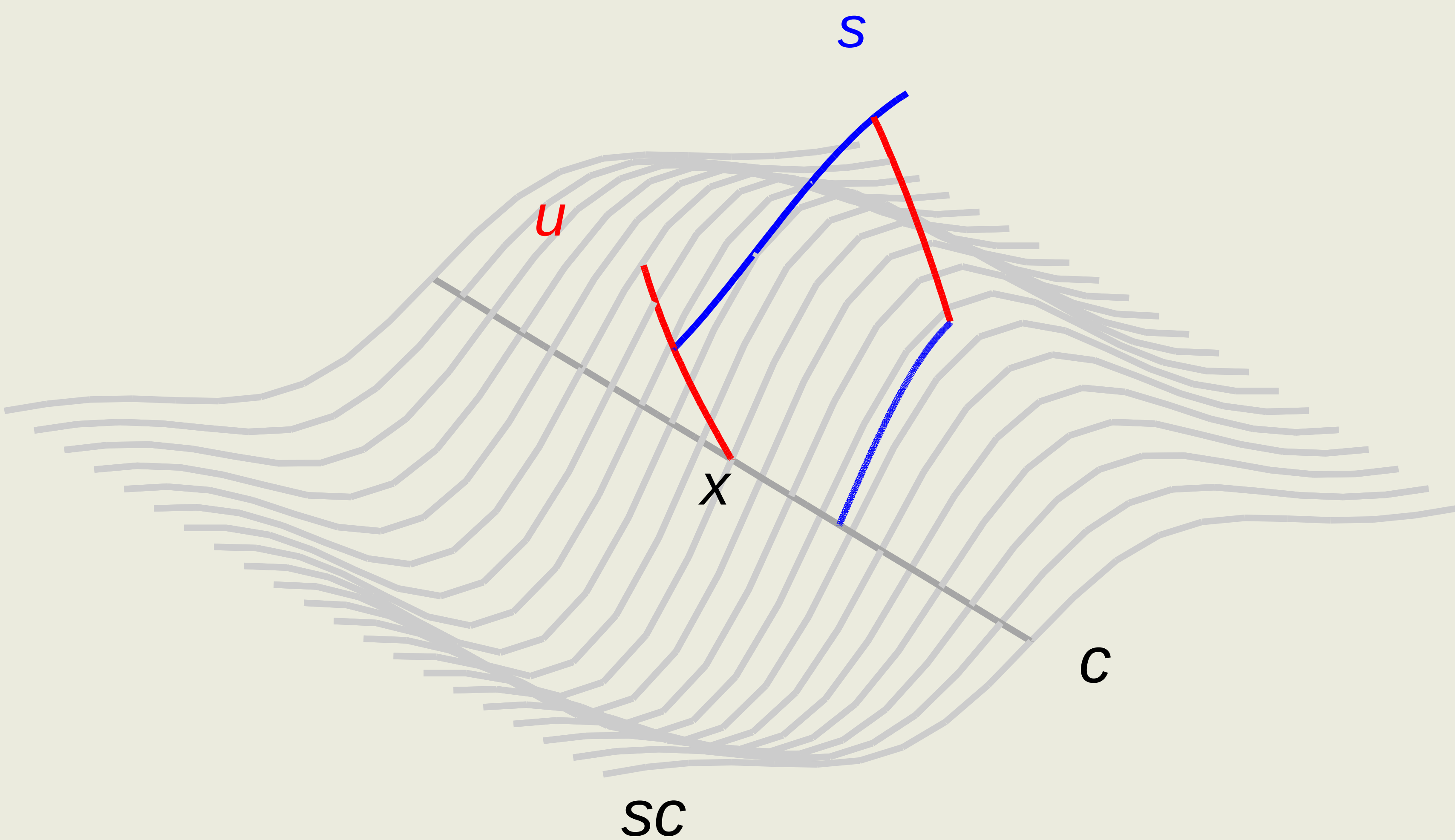
- $U = \{x \; : \; AC(x)$ is open $\}$

accessibility classes

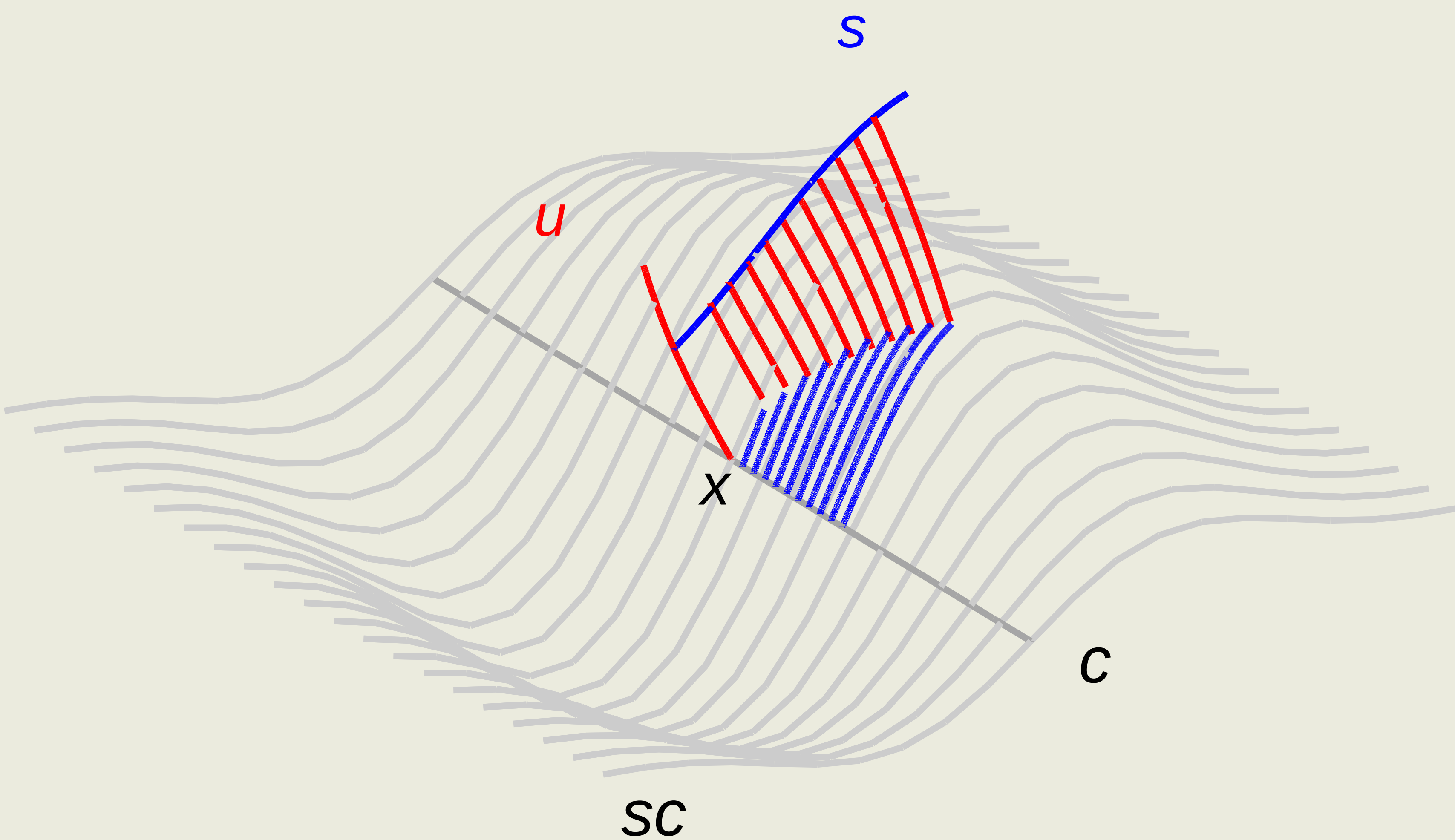
$AC(x) =$ accessibility class of x

- $U = \{x \; : \; AC(x)$ is open $\}$
- $\Gamma = M \setminus U$

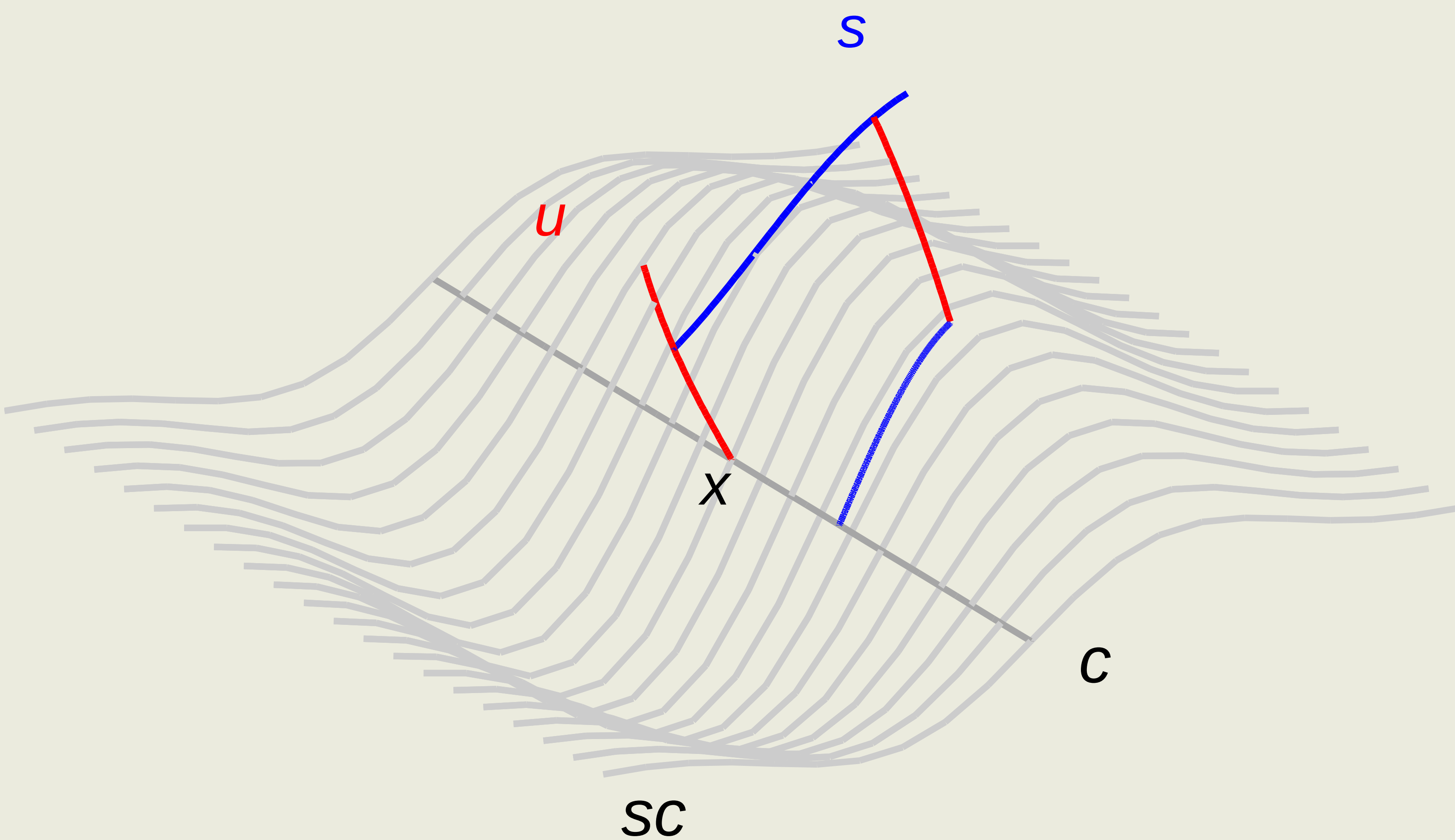
$AC(x)$ is open



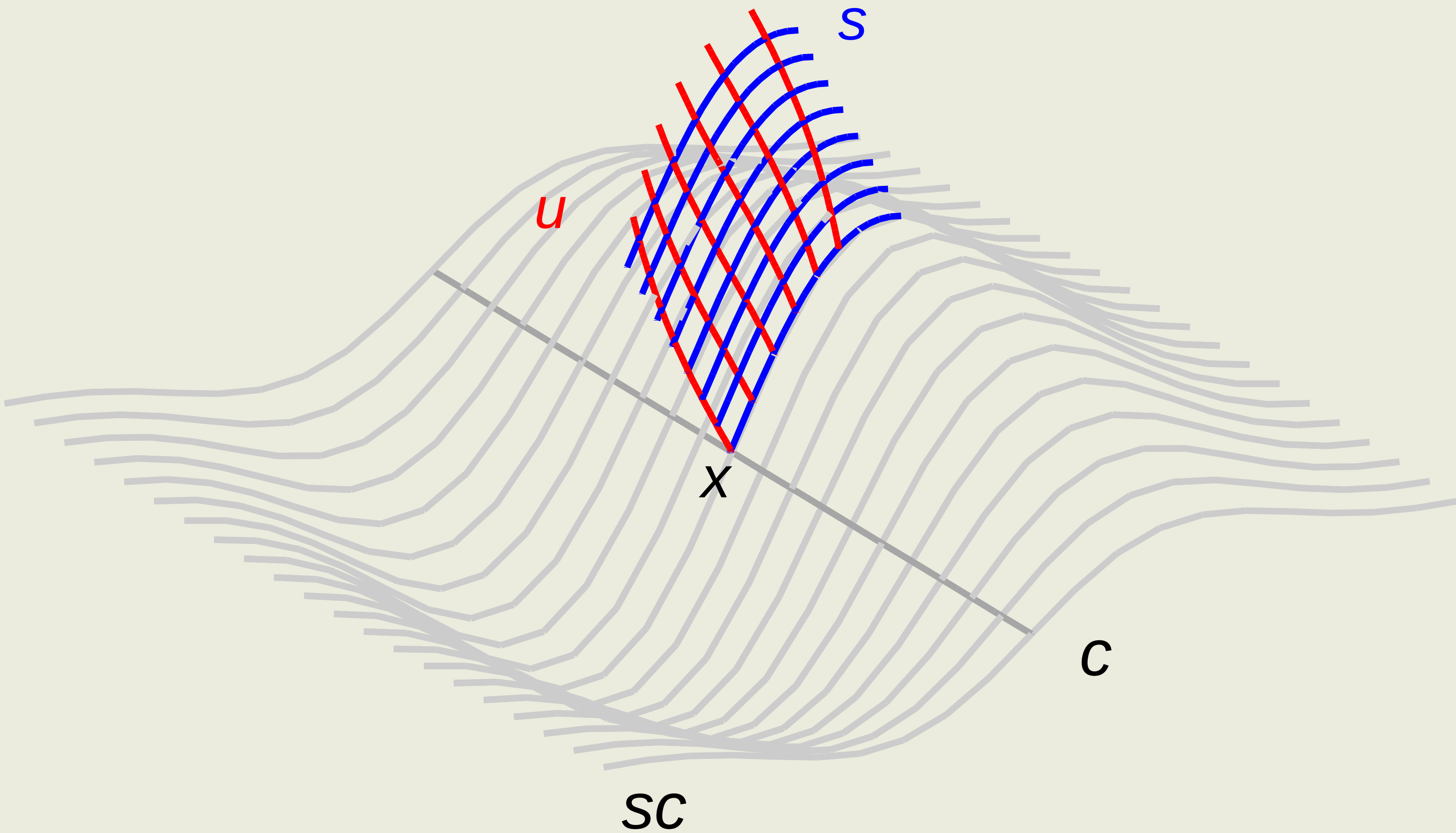
$AC(x)$ is open



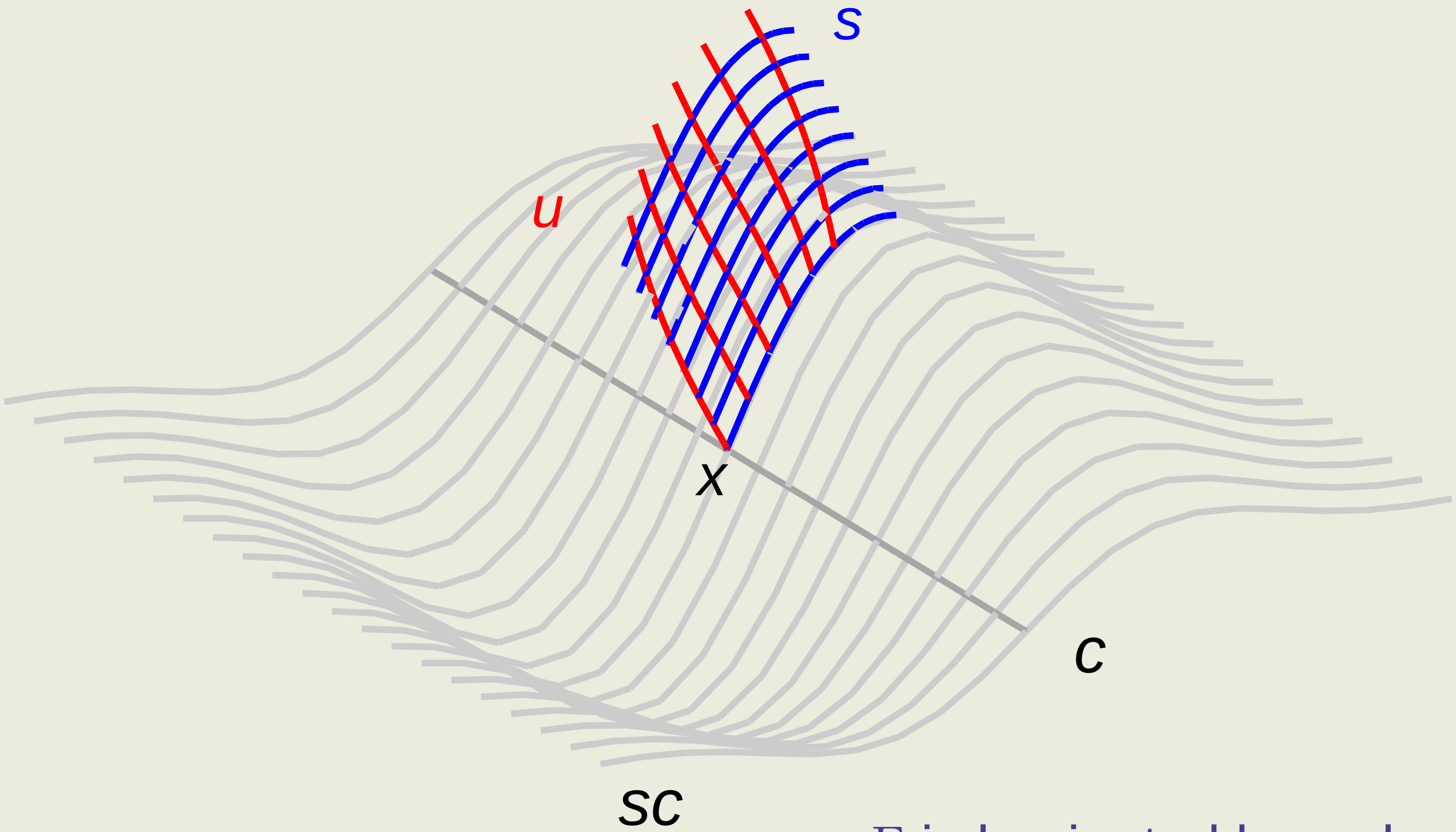
$AC(x)$ is open



$AC(x)$ is not open - $x \in \Gamma$



$AC(x)$ is not open - $x \in \Gamma$



$\Rightarrow \Gamma$ is laminated by su leaves

3 possibilities

- $\Gamma = \emptyset$

3 possibilities

- $\Gamma = \emptyset$
- $\emptyset \subsetneq \Gamma \subsetneq M$

3 possibilities

- $\Gamma = \emptyset$
- $\emptyset \subsetneq \Gamma \subsetneq M$
- $\Gamma = M$

3 possibilities

- $\Gamma = \emptyset$ $\Rightarrow f$ has the accessibility property
- $\emptyset \subsetneq \Gamma \subsetneq M$
- $\Gamma = M$

3 possibilities

- $\Gamma = \emptyset$ $\Rightarrow f$ has the accessibility property
- $\emptyset \subsetneq \Gamma \subsetneq M$
- $\Gamma = M$ $\Rightarrow M$ is foliated by su leaves

definitions

partially hyperbolic

$f : M \to M$ is partially hyperbolic if

$$TM = E^s \oplus E^c \oplus E^u \qquad \text{invariant}$$

where all $v^\sigma \in E^\sigma_x$ (unitary vectors) verify: $(\sigma = s, c, u)$

- $\|Tf_x v^s\| < 1 < \|Tf_x v^u\|$
- $\|Tf_x v^s\| < \|Tf_x v^c\| < \|Tf_x v^u\|$

accessibility

$f : M \to M$ has the accessibility property

if M is the unique s- and u-saturated set

accessibility class

the accessibility class of $x \in M$

is the minimal s- and u-saturated set containing x.

center bunching

f is center bunched if there are continuous functions γ and $\hat{\gamma}$ verifying, for all unitary $v^c \in E^c$

- $\gamma(x) \leq \|Tf_x v^c\| \leq \hat{\gamma}(x)$
- $\|Tf_x v^s\|^\theta < \frac{\gamma(x)}{\hat{\gamma}(x)}$

for some suitable θ

references - conjecture A

- M^n with center bunching ($\Rightarrow$ case $\dim E^c = 1$) (Burns, Wilkinson '05)

*

references - conjecture A

- M^n with center bunching ($\Rightarrow$ case $\dim E^c = 1$)
  (Burns, Wilkinson '05)
- M^n with $\dim E^c = 1$
  (R. Hertz, R. Hertz, Ures '05)

*

references - conjecture B

- Stable accessibility is C^1 dense (Dolgopyat, Wilkinson '03)

*

references - conjecture B

- Stable accessibility is C^1 dense (Dolgopyat, Wilkinson '03)
- Accessibility is open if $\dim E^c = 1$ (Didier '03)

*

references - conjecture B

- Stable accessibility is C^1 dense (Dolgopyat, Wilkinson ’03)
- Accessibility is open if $\dim E^c = 1$ (Didier ’03)
- Accessibility is C^∞ dense if $\dim E^c = 1$ & f conservative (R.Hertz, R. Hertz, Ures ’05)

*

references - conjecture B

- Stable accessibility is C^1 dense (Dolgopyat, Wilkinson '03)
- Accessibility is open if $\dim E^c = 1$ (Didier '03)
- Accessibility is C^∞ dense if $\dim E^c = 1$ & f conservative (R.Hertz, R. Hertz, Ures '05)
- Accessibility is C^∞ dense if $\dim E^c = 1$ (Burns, R.Hertz, R. Hertz, Talitskaya, Ures '05)

*