C^1 Pugh Shub conjecture for center dimension 2

F. Rodriguez Hertz
M.A. Rodriguez Hertz
A. Tahzibi
R. Ures

question

How frequent is ergodicity?

stable ergodicity

$f : M \to M$ is stably ergodic if there is $\mathcal{U} \subset C^1$ open s.t.

stable ergodicity

$f : M \to M$ is stably ergodic if there is $\mathcal{U} \subset C^1$ open s.t.

$$f \in \mathcal{U} \cap \mathrm{Diff}^2_m(M) \subset \text{ ergodic}$$

question

sufficient conditions for stable ergodicity?

examples

- Anosov diffeomorphisms (1967)

examples

- Anosov diffeomorphisms (1967)
- time-one map of the geodesic flow of a surface with $\kappa < 0$ (1994)

partial hyperbolicity

$f : M \to M$ is partially hyperbolic if

partial hyperbolicity

$f : M \to M$ is partially hyperbolic if

$$TM = E^s \oplus E^c \oplus E^u$$

partial hyperbolicity

$f : M \to M$ is partially hyperbolic if

$$TM \quad = \quad \underset{\substack{\uparrow \\ \text{contracting}}}{E^s} \quad \oplus \quad E^c \quad \oplus \quad \underset{\substack{\uparrow \\ \text{expanding}}}{E^u}$$

ergodic Pugh-Shub conjecture (1995)

Partial hyperbolicity is essentially a sufficient condition for stable ergodicity.

C^r ergodic Pugh-Shub conjecture (1995)

Stable ergodicity is C^r dense among partially hyperbolic diffeomorphisms.

Pugh-Shub program

CONJECTURE A (Essential) accessibility $\Rightarrow$ ergodicity

Pugh-Shub program

CONJECTURE A (Essential) accessibility $\Rightarrow$ ergodicity

CONJECTURE B Stable accessibility is C^r dense among partially hyperbolic diffeomorphisms, volume preserving or not

advances - conjecture A

BW (2006) (Essential) accessibility $\Rightarrow$ ergodicity

center bunching

advances - conjecture A

BW (2006) (Essential) accessibility $\Rightarrow$ ergodicity
center bunching

BDP (2002) (Essential) accessibility $\Rightarrow$ ergodicity
$\lambda(x, v^c) > 0$ over a positive measure set

advances - conjecture B

RHRHU (2007) Stable accessibility is C^r dense among volume preserving partially hyperbolic diffeomorphisms with $\dim E^c = 1$.

advances - conjecture B

RHRHU (2007) Stable accessibility is C^r dense among volume preserving partially hyperbolic diffeomorphisms with $\dim E^c = 1$.

BRHRHTU (2007) Stable accessibility is C^r dense among partially hyperbolic diffeomorphisms with $\dim E^c = 1$.

advances - conjecture B

RHRHU (2007) Stable accessibility is C^r dense among volume preserving partially hyperbolic diffeomorphisms with $\dim E^c = 1$.

BRHRHTU (2007) Stable accessibility is C^r dense among partially hyperbolic diffeomorphisms with $\dim E^c = 1$.

DW (2003) Stable accessibility is C^1 dense among partially hyperbolic diffeomorphisms.

Theorem - RH.RH.T.U (2007)

Stable ergodicity is C^1 dense among C^2 partially hyperbolic diffeomorphisms with $\dim E^c = 2$.

sketch of the proof

for $f \in C^2$ partially hyperbolic with $\dim E^c = 2$, either

- f approximated by a stably ergodic diffeomorphism

sketch of the proof

for $f \in C^2$ partially hyperbolic with $\dim E^c = 2$, either

- f approximated by a stably ergodic diffeomorphism
- f approximated by a diffeo g satisfying ① ② ③ ④

sketch of the proof

for $f \in C^2$ partially hyperbolic with $\dim E^c = 2$, either

- f approximated by a stably ergodic diffeomorphism
- f approximated by a diffeo g satisfying (1)(2)(3)(4)

$\lambda_g^+(x) \geq \lambda_g^-(x)$ Lyapunov exponents of $g|_{E^c}$

sketch of the proof

① g is stably accessible

sketch of the proof

1. g is stably accessible
2. $E^c_g = E^-_g \oplus E^u_g$ non-trivial dominated splitting

sketch of the proof

1. g is stably accessible
2. $E^c_g = E^-_g \oplus E^u_g$ non-trivial dominated splitting
3. $\int \lambda^-_g dm < 0$ and $\int \lambda^+_g dm > 0$

sketch of the proof

1. g is stably accessible
2. $E_g^c = E_g^- \oplus E_g^u$ non-trivial dominated splitting
3. $\int \lambda_g^- dm < 0$ and $\int \lambda_g^+ dm > 0$
4. $\exists\ \mathrm{Bl}^{cs}(p_{s+1})$ and $\mathrm{Bl}^{cu}(p_{s+1})$

sketch of the proof

1. g is stably accessible
2. $E^c_g = E^-_g \oplus E^u_g$ non-trivial dominated splitting
3. $\int \lambda^-_g dm < 0$ and $\int \lambda^+_g dm > 0$
4. $\exists \ \mathrm{Bl}^{cs}(p_{s+1})$ and $\mathrm{Bl}^{cu}(p_{s+1})$

$\Rightarrow f$ is stably ergodic

step ①

① g stably accessible

Stable accessibility is dense (DW)

step ②

② $E^c_g = E^-_g \oplus E^+_g$ non-trivial dominated splitting

step ②

② $E^c_g = E^-_g \oplus E^+_g$ non-trivial dominated splitting

Bochi-Viana type argument

step ②

② $E_g^c = E_g^- \oplus E_g^+$ non-trivial dominated splitting

Bochi-Viana type argument

$$\int_M \lambda_g^- dm \;\geq\; \int_M \lambda_f^- dm + \int_{NDS} \frac{\lambda_f^+ - \lambda_f^-}{2} dm - \varepsilon$$

step ②

② $E^c_g = E^-_g \oplus E^+_g$ non-trivial dominated splitting

Bochi-Viana type argument

$$\int_M \lambda^-_g dm \geq \int_M \lambda^-_f dm + \int_{NDS} \frac{\lambda^+_f - \lambda^-_f}{2} dm - \varepsilon$$
$$\geq \int_M \frac{\lambda^+_f + \lambda^-_f}{2} - \varepsilon$$

step ②

② $E_g^c = E_g^- \oplus E_g^+$ non-trivial dominated splitting

Bochi-Viana type argument

$$\begin{aligned}\int_M \lambda_g^- dm &\geq \int_M \lambda_f^- dm + \int_{NDS} \frac{\lambda_f^+ - \lambda_f^-}{2} dm - \varepsilon \\ &\geq \int_M \frac{\lambda_f^+ + \lambda_f^-}{2} - \varepsilon > 0\end{aligned}$$

step ②

② $E_g^c = E_g^- \oplus E_g^+$ non-trivial dominated splitting

Bochi-Viana type argument

$$\begin{aligned} \int_M \lambda_g^- dm &\geq \int_M \lambda_f^- dm + \int_{NDS} \frac{\lambda_f^+ - \lambda_f^-}{2} dm - \varepsilon \\ &\geq \int_M \frac{\lambda_f^+ + \lambda_f^-}{2} - \varepsilon > 0 \end{aligned}$$

note

$$\int_M (\lambda_f^+ + \lambda_f^-) dm = \int_M \log Jac(Tf|_{E_f^c}) dm > 0 \quad \text{(BB)}$$

step ③

③ $\int \lambda_g^- dm < 0$ and $\int \lambda_g^+ dm > 0$

step ③

③ $\int \lambda_g^- dm < 0$ and $\int \lambda_g^+ dm > 0$

Baraviera-Bonatti implies

$$\int_M \lambda_g^+ dm \neq 0$$

and

$$\int_M \lambda_g^- dm \neq 0$$

step ③

③ $\int \lambda_g^- dm < 0$ and $\int \lambda_g^+ dm > 0$

Baraviera-Bonatti implies

$$\int_M \lambda_g^+ dm \neq 0$$

and

$$\int_M \lambda_g^- dm \neq 0$$

either $\int_M \lambda_g^+ dm . \int_M \lambda_g^- dm > 0$

step ③

③ $\int \lambda_g^- dm < 0$ and $\int \lambda_g^+ dm > 0$

Baraviera-Bonatti implies

$$\int_M \lambda_g^+ dm \neq 0$$

and

$$\int_M \lambda_g^- dm \neq 0$$

either $\int_M \lambda_g^+ dm . \int_M \lambda_g^- dm > 0$ (BDP)

step ③

③ $\int \lambda_g^- dm < 0$ and $\int \lambda_g^+ dm > 0$

Baraviera-Bonatti implies

$$\int_M \lambda_g^+ dm \neq 0$$

and

$$\int_M \lambda_g^- dm \neq 0$$

either $\int_M \lambda_g^+ dm . \int_M \lambda_g^- dm > 0$ (BDP)

or ③

step ④

④a g has hyperbolic periodic points p_s, p_{s+1}, p_{s+2}

- E_f^+ and E_f^- are not hyperbolic

step 4

4a g has hyperbolic periodic points p_s, p_{s+1}, p_{s+2}

- E_f^+ and E_f^- are not hyperbolic
- conservative ergodic closing lemma $\to \lambda_g^+(p) \sim 0$ and $\lambda_g^-(q) \sim 0$

step (4)

(4a) g has hyperbolic periodic points p_s, p_{s+1}, p_{s+2}

- E_f^+ and E_f^- are not hyperbolic
- conservative ergodic closing lemma $\longrightarrow \lambda_g^+(p) \sim 0$ and $\lambda_g^-(q) \sim 0$
- conservative Franks' lemma $\longrightarrow \lambda_g^-(p_s) > 0$ and $\lambda_g^+(p_{s+2}) < 0$

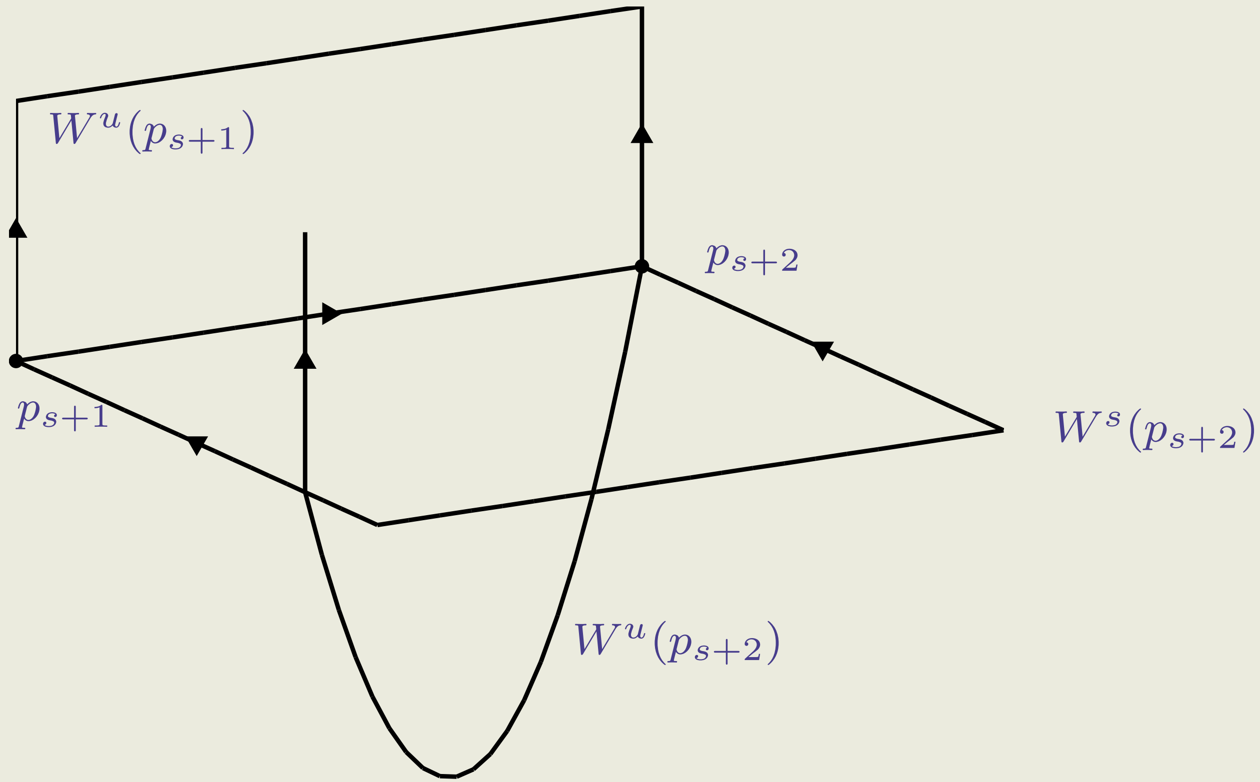
step (4)

(4a) g has hyperbolic periodic points p_s, p_{s+1}, p_{s+2}

- E_f^+ and E_f^- are not hyperbolic
- conservative ergodic closing lemma $\longrightarrow \lambda_g^+(p) \sim 0$ and $\lambda_g^-(q) \sim 0$
- conservative Franks' lemma $\longrightarrow \lambda_g^-(p_s) > 0$ and $\lambda_g^+(p_{s+2}) < 0$
- and $\lambda_g^+(p_{s+1}).\lambda_g^-(p_{s+1}) < 0$

step 4

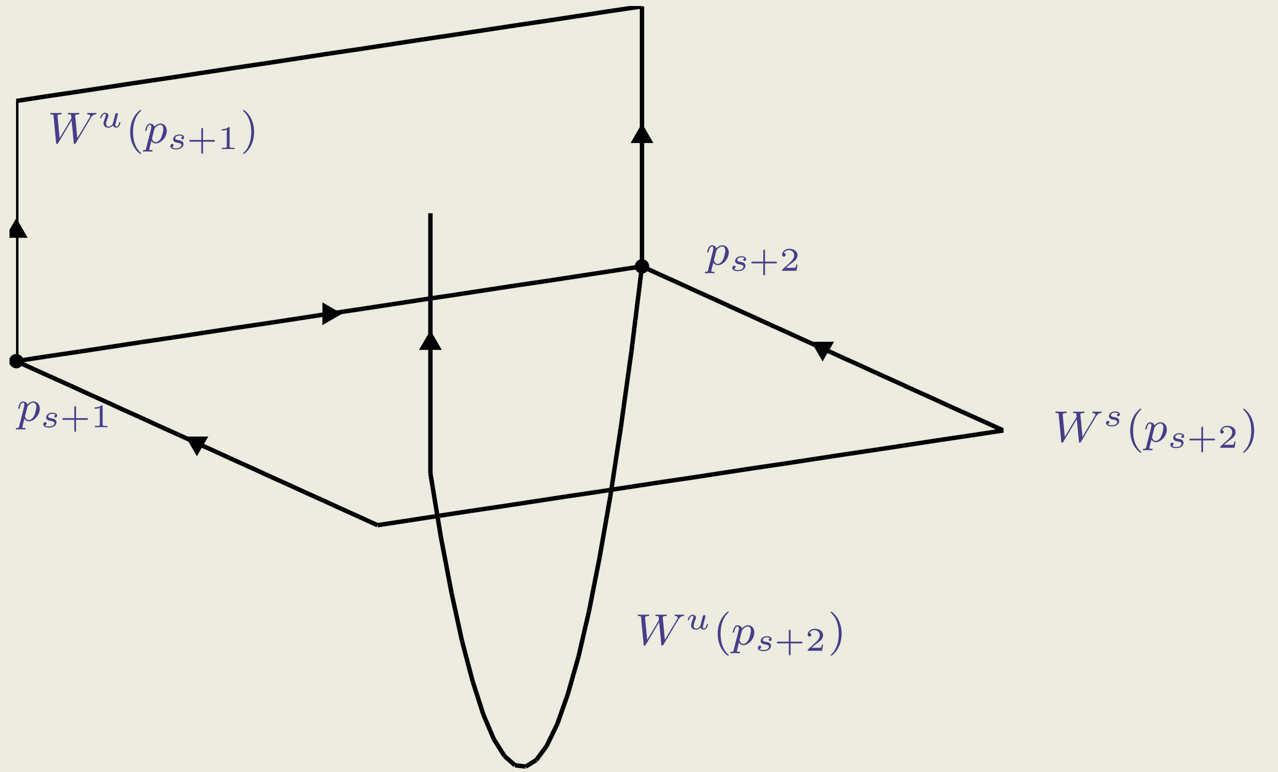
(4b) $\exists \, \mathrm{Bl}^{cu}(p_{s+1})$



conservative connecting lemma

step ④

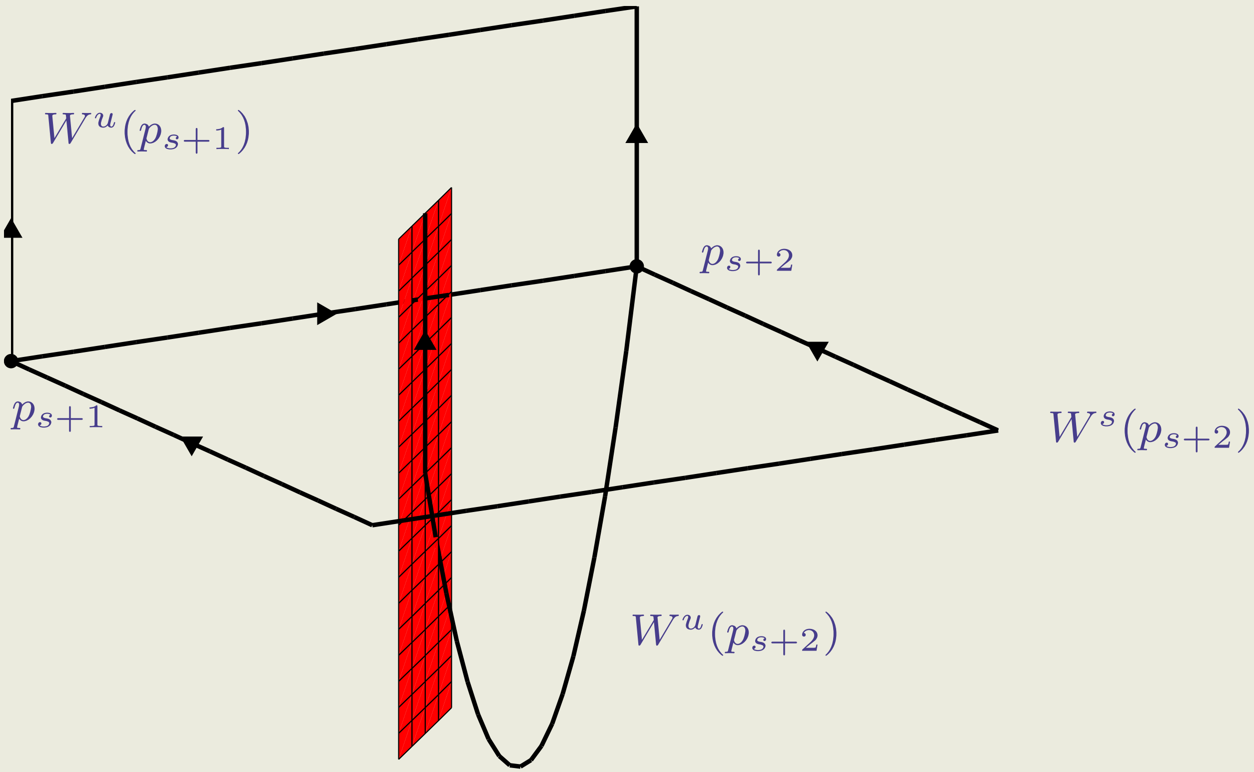
④b $\exists\ \mathrm{Bl}^{cu}(p_{s+1})$



creation of blender

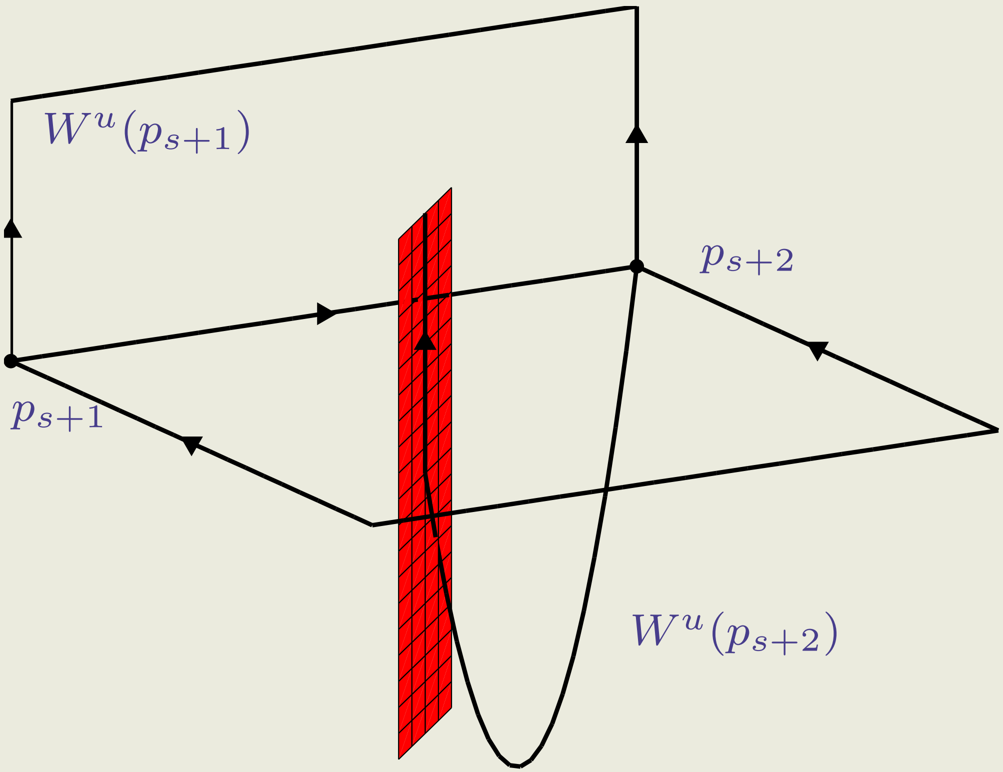
step ④

④b $\exists\ \mathrm{Bl}^{cu}(p_{s+1})$



every vertical strip $\cap\ W^s(p_{s+1})$

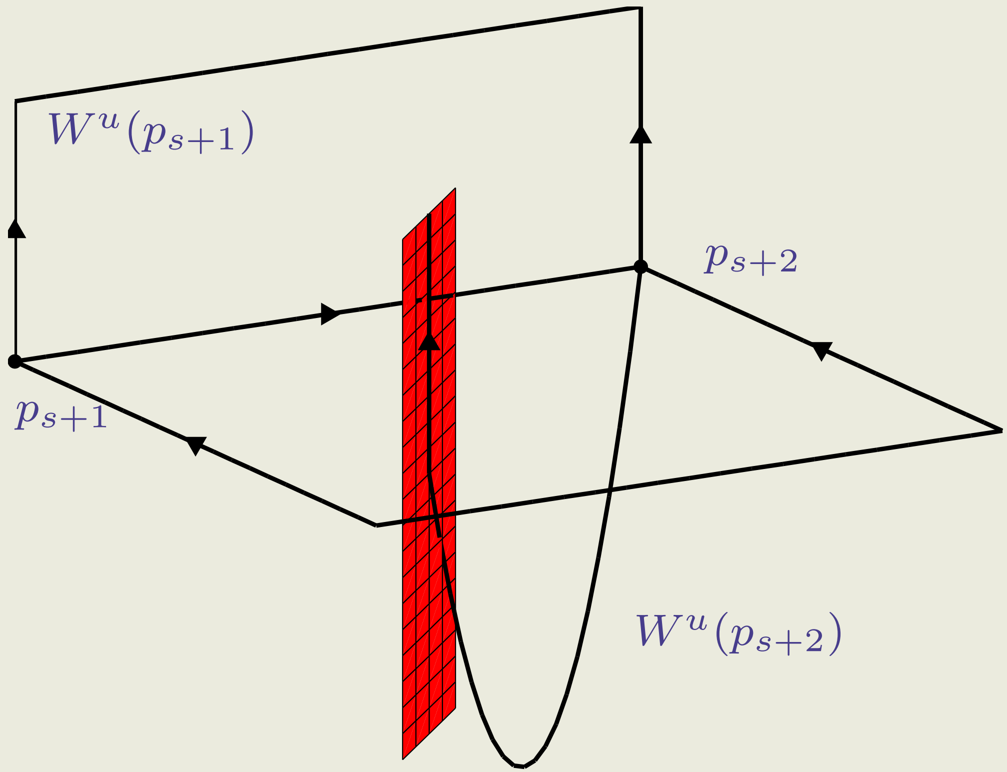
ergodicity



$$A^+ = x : \lambda^+(x) > 0$$

$$A^- = x : \lambda^-(x) < 0$$

ergodicity

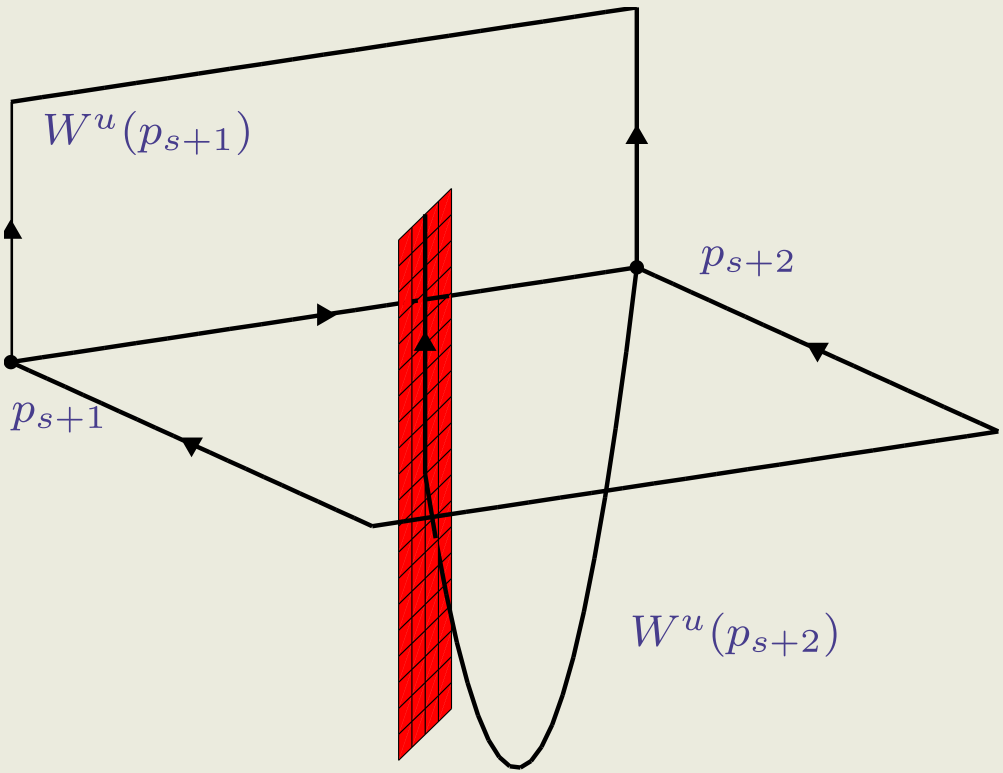


$$A^+ = x : \lambda^+(x) > 0$$

$$A^- = x : \lambda^-(x) < 0$$

$$m(A^+ \cup A^-) = 1$$

ergodicity



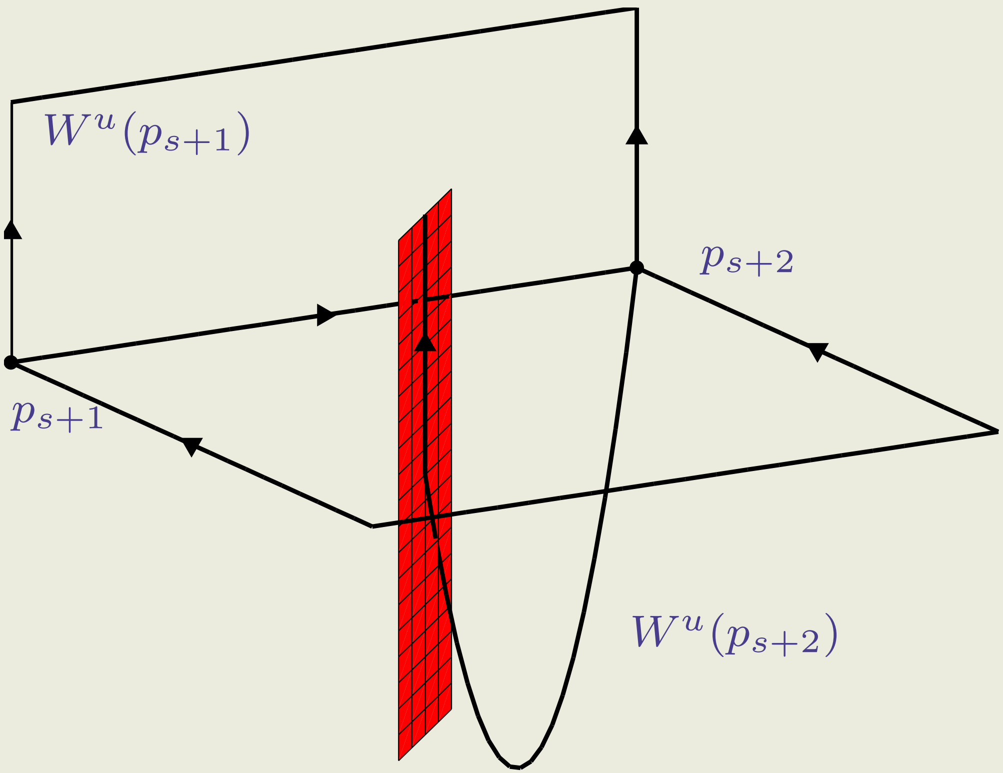
$$A^+ = x : \lambda^+(x) > 0$$

$$A^- = x : \lambda^-(x) < 0$$

$$m(A^+ \cup A^-) = 1$$

$$A^+ \subset B^s(p_{s+1})$$

ergodicity



$$A^+ = x : \lambda^+(x) > 0$$

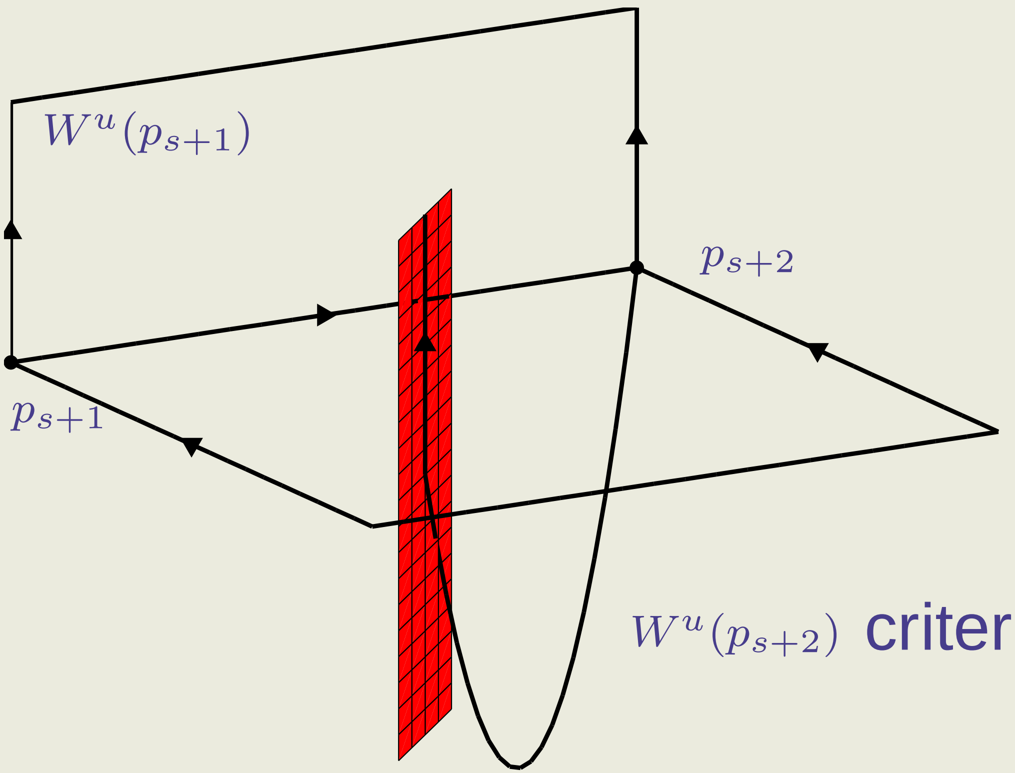
$$A^- = x : \lambda^-(x) < 0$$

$$m(A^+ \cup A^-) = 1$$

$$A^+ \subset B^s(p_{s+1})$$

$$A^- \subset B^u(p_{s+1})$$

ergodicity



$$A^+ = x : \lambda^+(x) > 0$$

$$A^- = x : \lambda^-(x) < 0$$

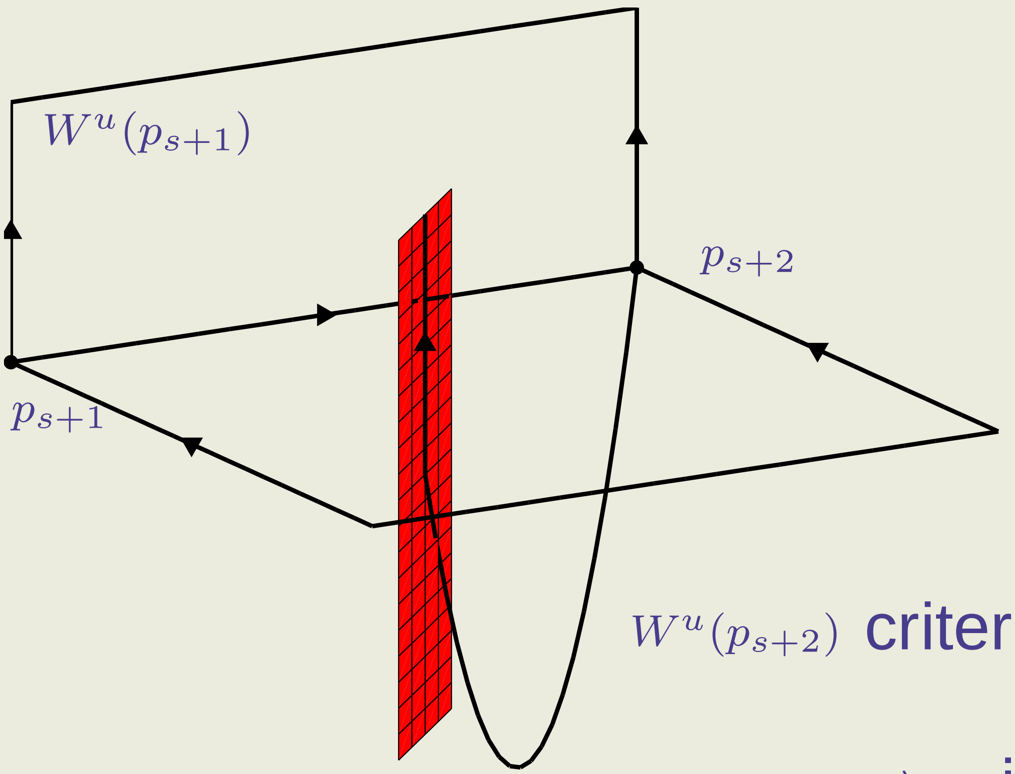
$$m(A^+ \cup A^-) = 1$$

$$A^+ \subset B^s(p_{s+1})$$

$$A^- \subset B^u(p_{s+1})$$

criterion applies

ergodicity



$$A^+ = x : \lambda^+(x) > 0$$

$$A^- = x : \lambda^-(x) < 0$$

$$m(A^+ \cup A^-) = 1$$

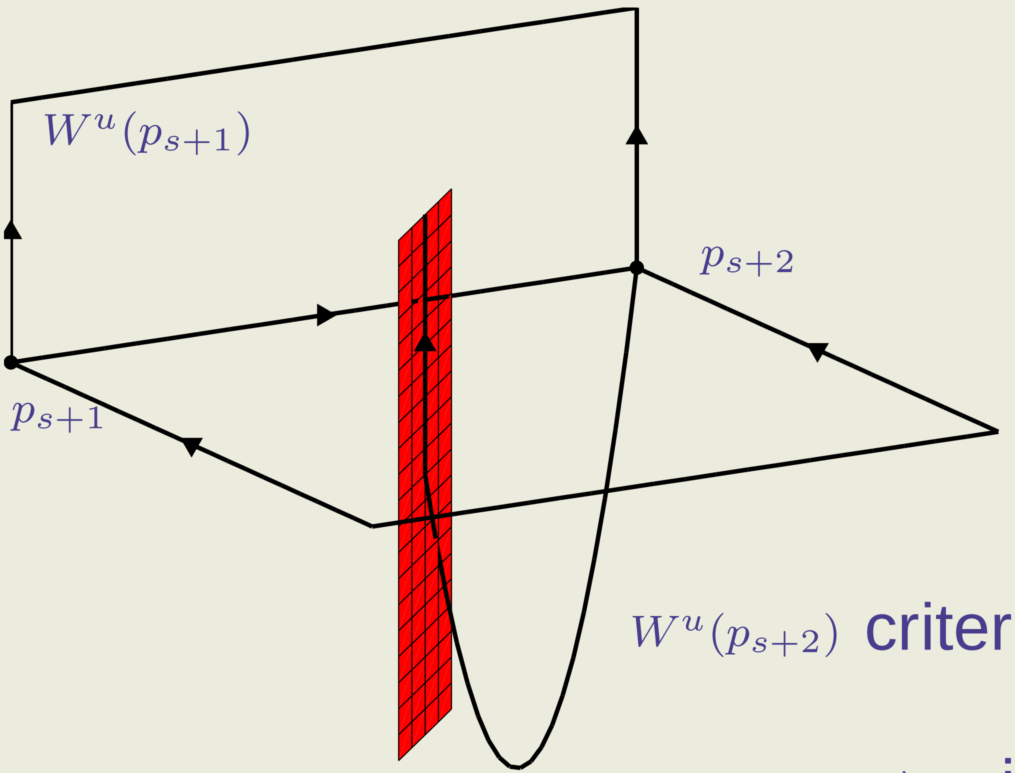
$$A^+ \subset B^s(p_{s+1})$$

$$A^- \subset B^u(p_{s+1})$$

criterion applies

$\Rightarrow g$ is ergodic

ergodicity



$$A^+ = x : \lambda^+(x) > 0$$

$$A^- = x : \lambda^-(x) < 0$$

$$m(A^+ \cup A^-) = 1$$

$$A^+ \subset B^s(p_{s+1})$$

$$A^- \subset B^u(p_{s+1})$$

criterion applies

$\Rightarrow g$ is ergodic $\square$