Partially hyperbolic systems with $\dim E^c = 1$

I. Stable ergodicity

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Question

How frequent is ergodicity among conservative systems?

Conjecture

CONJECTURE [PUGH,SHUB] *Stable ergodicity is* C^r *open and dense among conservative partially hyperbolic systems,* $r \in [2, \infty]$

Main theorem

MAIN THEOREM: *Stable ergodicity is* C^r *open and dense among conservative partially hyperbolic systems with* $\dim E^c = 1$, $r \in [2, \infty]$

Pugh-Shub program

THEOREM A *Accessibility is* C^r *open and dense among conservative partially hyperbolic conservative systems with* $\dim E^c = 1$

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THEOREM A *Accessibility is C^r open and dense among conservative partially hyperbolic conservative systems with* $\dim E^c = 1$

THEOREM B *Essential accessibility* $\Rightarrow$ *ergodicity in* $\mathcal{PH}^2_m(M)$ *if* $\dim E^c = 1$

Pugh-Shub program

THEOREM A *Accessibility is C^r open and dense among conservative partially hyperbolic conservative systems with $\dim E^c = 1$*

THEOREM B *Essential accessibility $\Rightarrow$ ergodicity in $\mathcal{PH}^2_m(M)$ if $\dim E^c = 1$*

THEOREM (BURNS,WILKINSON(2005))
Essential accessibility $\Rightarrow$ ergodicity in $\mathcal{PH}^2_m(M)$ if f is center bunched

Proof of theorem A

THEOREM (DIDIER (2003)) *Accessibility is C^1-open among conservative partially hyperbolic systems with* $\dim E^c = 1$

Proof of theorem A

PROPOSITION A.1 *C^r densely in $\mathcal{PH}^r_m(M)$, either:*

- *f has the accessibility property or*
- *$E^s \oplus E^u$ is integrable and $\mathcal{P}er(f) = \emptyset$*

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PROPOSITION A.2 *$(*)$ is nowhere dense in $\mathcal{PH}^r_m(M)$*

Proof of theorem B - Hopf argument

$$\phi_\pm(x) = \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} \phi(f^{\pm n}(x)) \xrightarrow{?} \int_M \phi dm \qquad \forall \phi \in C^0(M,\mathbb{R})$$

m-ae x

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$\Longrightarrow$ $S(c)\cap U(c)$ is essentially s- and u- saturated

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Hopf contex ▶ $S(c) \cap U(c)$ density points is an s- and u- saturated set

▼

$m(S(c) \cap U(c)) = 0$ or 1

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m-ae x

- $S(c) = \phi_+^{-1}[c,\infty)$ is s-saturated
- $U(c) = \phi_-^{-1}[c,\infty)$ is u-saturated $\Longrightarrow$ $S(c) \cap U(c)$ is essentially s- and u- saturated
- $m(S(c) \triangle U(c)) = 0$

this context ▶ $S(c) \cap U(c)$ density points is an s- and u- saturated set

▼

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Proof of theorem B

X is essentially s-saturated
and essentially u-saturated $\Longrightarrow$ the set of
Lebesgue density points
of X is s-saturated

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PROPOSITION B.1 *X is essentially s-saturated $\Longrightarrow$ the set of SUC_n-density points of X is s-saturated*

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DEFINITION *$\mathscr{M}$ family of measurable sets*
$\{B_n\}$ and $\{C_n\}$ are Vitali equivalent over $\mathscr{M}$ if

$$B_n\text{-density points}(X) \overset{\substack{\text{pointwise}\\ \downarrow}}{\doteq} C_n\text{-density points}(X) \qquad \forall\, X \in \mathscr{M}$$

Theorem A

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definitions

partially hyperbolic

$f : M \to M$ is partially hyperbolic if

$$TM = E^s \oplus E^c \oplus E^u \qquad \text{invariant}$$

where all $v^\sigma \in E^\sigma_x$ (unitary vectors) verify: $(\sigma = s, c, u)$

- $\|Df(x)v^s\| < \|Df(x)v^c\| < \|Df(x)v^u\|$
- $\|Df(x)v^s\| < 1 < \|Df(x)v^u\|$

accessibility

$f : M \to M$ has the accessibility property

if M is the unique non-void s and u-saturated set

essential accessibility

$f : M \to M$ has the essential accessibility property

if s and u-saturated sets have either full or null measure

essentially σ-saturated

X is an essentially σ-saturated set

if there exists a σ-saturated set X_σ such that

$$m(X \triangle X_\sigma) = 0$$

center bunching

f is center bunched if there are continuous functions γ and $\hat{\gamma}$ verifying, for all unitary $v^c \in E^c$

- $\gamma(x) \leq \|Df(x)v^c\| \leq \hat{\gamma}(x)$
- $\|Df(x)v^s\|^\theta < \frac{\gamma(x)}{\hat{\gamma}(x)}$

for some suitable θ

density points

- x is a Lebesgue density point of a measurable set X if

$$\lim_{\varepsilon \to 0} \frac{m(X \cap B_\varepsilon(x))}{m(B_\varepsilon(x))} = 1$$

- x is a B_n-density point of X if

$$\lim_{n \to \infty} \frac{m(X \cap B_n(x))}{m(B_n(x))} = 1$$

where $x \mapsto \{B_n(x)\}_{n \in \mathbb{N}}$ is a countable neighborhood basis system