Topological dynamics

Symbolic dynamics

Jana Rodriguez Hertz

Universidad de la República - Uruguay

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the space Σ^+

Σ^+

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$$\Sigma^+ = \{0,1\}^{\mathbb{N}}$$

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x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	...
1	0	1	1	0	0	1	0	0	0	1	1	0	...

the space Σ^+

$$\Sigma^+ = \{0,1\}^{\mathbb{N}}$$

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	...
1	0	1	1	0	0	1	0	0	0	1	1	0	...
0	1	1	1	1	0	1	0	1	0	0	0	1	...

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x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	...
1	0	1	1	0	0	1	0	0	0	1	1	0	...
0	1	1	1	1	0	1	0	1	0	0	0	1	...
0	0	1	0	1	1	0	0	0	1	1	0	1	...

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x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	...
1	0	1	1	0	0	1	0	0	0	1	1	0	...
0	1	1	1	1	0	1	0	1	0	0	0	1	...
0	0	1	0	1	1	0	0	0	1	1	0	1	...
1	0	1	1	0	0	1	1	1	0	0	1	1	...

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x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	...
1	0	1	1	0	0	1	0	0	0	1	1	0	...
0	1	1	1	1	0	1	0	1	0	0	0	1	...
0	0	1	0	1	1	0	0	0	1	1	0	1	...
1	0	1	1	0	0	1	1	1	0	0	1	1	...
⋮						⋮							

a metric on Σ^+

We can define a metric on Σ^+:

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$$d(\underline{x}, \underline{y}) =$$

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$$d(\underline{x}, \underline{y}) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{3^{n+1}}$$

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Proposition

- (Σ^+, d) is a compact metric space

a metric on Σ^+

We can define a metric on Σ^+:

$$d(\underline{x}, \underline{y}) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{3^{n+1}}$$

Proposition

- (Σ^+, d) is a compact metric space
- $d(\underline{x}, \underline{y}) < {}^1/_{3^{n+1}} \quad \Leftrightarrow \quad x_i = y_i$ for $i = 0, \ldots, n$

example

example

points in $B(\underline{1}, {}^1\!/_{36})$

example

example

points in $B(\underline{1}, {}^{1}/_{36})$

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	...
1	1	1	1	1	1	1	0	0	0	1	1	0	...

example

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points in $B(\underline{1}, {}^1/_{36})$

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	...
1	1	1	1	1	1	1	0	0	0	1	1	0	...
1	1	1	1	1	1	1	0	1	0	0	0	1	...

example

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points in $B(\underline{1}, {}^1/_{36})$

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	...
1	1	1	1	1	1	1	0	0	0	1	1	0	...
1	1	1	1	1	1	1	0	1	0	0	0	1	...
1	1	1	1	1	1	0	0	0	1	1	0	1	...

example

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points in $B(\underline{1}, {}^{1}\!/_{36})$

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	...
1	1	1	1	1	1	1	0	0	0	1	1	0	...
1	1	1	1	1	1	1	0	1	0	0	0	1	...
1	1	1	1	1	1	0	0	0	1	1	0	1	...
1	1	1	1	1	1	1	1	1	0	0	1	1	...

example

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points in $B(\underline{1}, {}^{1}/_{36})$

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	...
1	1	1	1	1	1	1	0	0	0	1	1	0	...
1	1	1	1	1	1	1	0	1	0	0	0	1	...
1	1	1	1	1	1	0	0	0	1	1	0	1	...
1	1	1	1	1	1	1	1	1	0	0	1	1	...
⋮						⋮							

the shift transformation

Definition (the shift transformation)

the shift transformation $\sigma : \Sigma^+ \to \Sigma^+$

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is defined by

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is defined by

$$[\sigma(\underline{x})]_n = x_{n+1}$$

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$\underline{x}$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	...
$\underline{x}$	1	0	1	1	0	0	1	0	0	0	1	1	...

example

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$\underline{x}$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	...
$\underline{x}$	1	0	1	1	0	0	1	0	0	0	1	1	...
$\sigma(\underline{x})$	0	1	1	0	0	1	0	0	0	1	1	0	...

example

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$\underline{x}$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	...
$\underline{x}$	1	0	1	1	0	0	1	0	0	0	1	1	...
$\sigma(\underline{x})$	0	1	1	0	0	1	0	0	0	1	1	0	...
$\sigma^2(\underline{x})$	1	1	0	0	1	0	0	0	1	1	0	1	...

example

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$\sigma^2(\underline{x})$	1	1	0	0	1	0	0	0	1	1	0	1	...
$\sigma^3(\underline{x})$	1	0	0	1	0	0	0	1	1	0	1	1	...

example

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$\sigma^3(\underline{x})$	1	0	0	1	0	0	0	1	1	0	1	1	...
$\sigma^4(\underline{x})$	0	0	1	0	0	0	1	1	0	1	1	0	...

example

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$\underline{x}$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	$\dots$
$\underline{x}$	1	0	1	1	0	0	1	0	0	0	1	1	$\dots$
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$\sigma^3(\underline{x})$	1	0	0	1	0	0	0	1	1	0	1	1	$\dots$
$\sigma^4(\underline{x})$	0	0	1	0	0	0	1	1	0	1	1	0	$\dots$
$\sigma^5(\underline{x})$	0	1	0	0	0	1	1	0	1	1	0	1	$\dots$

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$\sigma^4(\underline{x})$	0	0	1	0	0	0	1	1	0	1	1	0	...
$\sigma^5(\underline{x})$	0	1	0	0	0	1	1	0	1	1	0	1	...
$\vdots$						$\vdots$							

fixed points

Definition (fixed point)

$\underline{x}$ is a fixed point if $\sigma(\underline{x}) = \underline{x}$

fixed points

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- 11

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periodic points

Definition (periodic point)

$\underline{x}$ is a periodic point if $\exists N \geq 0$ such that

$$o(\underline{x}) : \qquad \underline{x}, \sigma(\underline{x}), \sigma^2(\underline{x}), \ldots, \sigma^N(\underline{x}) = \underline{x}$$

periodic points of period 2

- $\underline{x}$ is a periodic point of period 2

periodic points of period 2

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- $\Rightarrow [\sigma^2(\underline{x})]_n = x_n$ for each $n \geq 0$

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- 4 cases

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- 4 cases
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  1. $x_0 x_1 = 00$
  2. $x_0 x_1 = 01$
  3. $x_0 x_1 = 10$

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- (2) 01

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- 4 cases
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  4. $x_0 x_1 = 11$
- (2) $010101\ldots$

periodic point are dense

periodic points are dense

the periodic points for the shift transformation are dense in Σ^+

transitivity

transitivity

the shift transformation is transitive

proof

proof

there is $\underline{x}$ with dense orbit:

proof

there is $\underline{x}$ with dense orbit:

$$\underline{x} =$$

proof

there is $\underline{x}$ with dense orbit:

$$\underline{x} = \boxed{0}$$

proof

there is $\underline{x}$ with dense orbit:

$\underline{x} = 0\ 1$

proof

there is $\underline{x}$ with dense orbit:

$$\underline{x} = 0\ 1\ 00$$

proof

there is $\underline{x}$ with dense orbit:

$\underline{x} = 0\ 1\ 00\ 01$

proof

there is $\underline{x}$ with dense orbit:

$$\underline{x} = 0\ 1\ 00\ 01\ 10$$

proof

there is $\underline{x}$ with dense orbit:

$$\underline{x} = 0\ 1\ 00\ 01\ 10\ 11$$

proof

there is $\underline{x}$ with dense orbit:

$$\underline{x} = 0\ 1\ 00\ 01\ 10\ 11\ 000$$

proof

there is $\underline{x}$ with dense orbit:

$$\underline{x} = 0\ 1\ 00\ 01\ 10\ 11\ 000\ 001$$

proof

there is $\underline{x}$ with dense orbit:

$$\underline{x} = 0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ 010$$

proof

there is $\underline{x}$ with dense orbit:

$\underline{x} =$ 0 1 00 01 10 11 000 001 010 011

proof

there is $\underline{x}$ with dense orbit:

$$\underline{x} = 0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ 010\ 011\ 100$$

proof

there is $\underline{x}$ with dense orbit:

$$\underline{x} = 0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ 010\ 011\ 100\ \ldots$$

proof

there is $\underline{x}$ with dense orbit:

$$\underline{x} = 0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ 010\ 011\ 100 \dots$$

$\underline{x}$ contains all finite sequences of 0's and 1's

the space Σ

Σ

the space Σ

$$\Sigma = \{0, 1\}^{\mathbb{Z}}$$

the space Σ

$$\Sigma = \{0, 1\}^{\mathbb{Z}}$$

...	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	...
...	0	1	1	0	0	1	0	0	0	1	1	...

the space Σ

$$\Sigma = \{0, 1\}^{\mathbb{Z}}$$

...	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	...
...	0	1	1	0	0	1	0	0	0	1	1	...
...	1	1	1	1	0	1	0	1	0	0	0	...

the space Σ

$$\Sigma = \{0, 1\}^{\mathbb{Z}}$$

...	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	...
...	0	1	1	0	0	1	0	0	0	1	1	...
...	1	1	1	1	0	1	0	1	0	0	0	...
...	0	1	0	1	1	0	0	0	1	1	0	...

the space Σ

$$\Sigma = \{0, 1\}^{\mathbb{Z}}$$

...	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	...
...	0	1	1	0	0	1	0	0	0	1	1	...
...	1	1	1	1	0	1	0	1	0	0	0	...
...	0	1	0	1	1	0	0	0	1	1	0	...
...	0	1	1	0	0	1	1	1	0	0	1	...

the space Σ

$$\Sigma = \{0, 1\}^{\mathbb{Z}}$$

...	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	...
...	0	1	1	0	0	1	0	0	0	1	1	...
...	1	1	1	1	0	1	0	1	0	0	0	...
...	0	1	0	1	1	0	0	0	1	1	0	...
...	0	1	1	0	0	1	1	1	0	0	1	...
	⋮					⋮						

a metric on Σ

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Proposition

- (Σ, d) is a compact metric space

a metric on Σ

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Proposition

- (Σ, d) is a compact metric space
- $d(\underline{x}, \underline{y}) < {}^{1}/_{3^{n+1}} \quad \Leftrightarrow \quad x_i = y_i$ for $|i| \leq n$

the shift transformation

Definition (the shift transformation)

the shift transformation $\sigma : \Sigma \to \Sigma$

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(just as with Σ_+)

properties of the two-sided shift

properties

the two sided shift:

properties of the two-sided shift

properties

the two sided shift:

- has dense periodic points

properties of the two-sided shift

properties

the two sided shift:

- has dense periodic points
- is transitive

fixed points

- $\underline{0} = 0000000000000000\dots$

fixed points

- $\underline{0} = 0000000000000000\ldots$
- $\underline{1} = 1111111111111111\ldots$

fixed points

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- $\underline{1} = 1111111111111111\ldots$

are they attracting?

fixed points

- $\underline{0} = 000000000000000\ldots$
- $\underline{1} = 111111111111111\ldots$

are they attracting? repelling?

the spaces Σ_n^+ and Σ_n

- Σ_n^+

the spaces Σ_n^+ and Σ_n

- $$\Sigma_n^+ = \{0, 1, \ldots, n-1\}^{\mathbb{N}}$$

the spaces Σ_n^+ and Σ_n

- $$\Sigma_n^+ = \{0, 1, \dots, n-1\}^{\mathbb{N}}$$
- $$\Sigma_n$$

the spaces Σ_n^+ and Σ_n

- $$\Sigma_n^+ = \{0, 1, \ldots, n-1\}^{\mathbb{N}}$$
- $$\Sigma_n = \{0, 1, \ldots, n-1\}^{\mathbb{Z}}$$

properties

properties

- Σ_n^+ has the same properties as $\Sigma^+ = \Sigma_2^+$

properties

properties

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- Σ_n has the same properties as $\Sigma = \Sigma_2$