Generalized phase-shifting algorithms: Error analysis and minimization of noise propagation

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Abstract

Phase-shifting is a technique for phase retrieval that requires a series of intensity measurements with certain phase-steps. The purpose of the present work is threefold: Firstly we present a new method for generating general phase-shifting algorithms with arbitrarily spaced phase-steps. Secondly, we study the conditions for which the phase-retrieval error due to phase-shift miscalibration can be minimized. Thirdly, we study the phase extraction from interferograms with additive random noise, and deduce the conditions to be satisfied for minimizing the phase-retrieval error. Algorithms with unevenly spaced phase-steps are discussed under linear phase-shift errors and additive Gaussian noise, and simulations are presented.

OCIS codes: (120.3180) Interferometry; (120.2650) Fringe Analysis.

1 Introduction

Phase-shifting interferometry (PSI) is a well-established technique for retrieving the phase of a test object (see e. g. [1-11] and the references therein). PSI requires a series of intensity measurements, $I_m(x, y)$ (with $m = 1, 2, ...M$), with certain phase-steps δ_m for retrieving the phase $\phi(x, y)$ of the sample. The interferograms are given by

$$I_m(x, y) = I_0(x, y)\left[1 + K(x, y)\cos\left(\phi(x, y) + \delta_m\right)\right] , \tag{1}$$

where (x, y) are Cartesian coordinates, $I_0(x, y)$ is the mean intensity and $K(x, y)$ is the contrast of the fringes.

In general, the algorithms for the phase recovering published in the literature can be written as a quotient of linear combinations of the I_m, i.e.

$$\tan(\phi) = \frac{\sum_{m=1}^{M} b_m I_m}{\sum_{m=1}^{M} a_m I_m} . \tag{2}$$

with given (real) coefficients a_m and b_m.

Expression (2) presupposes that the phase-steps are known. Most algorithms are deduced under the hypothesis that the δ_m are evenly spaced in the interval $[0, 2\pi]$, e.g. $\delta_m = (m-1)2\pi / N$. In the praxis, however, this requirement is often difficult to exactly meet because of the effective reference phases are determined not only by the phase shifter but also by any other events that change the relative optical path difference. Thus, the actual values of δ_m are only approximately known *a priori* [12], or they are to be determined *a posteriori* by different methods from the experimentally obtained interferograms (see e.g. [13,14]). In general, the actual (measured) phase-steps will not coincide with their nominal values and will not be evenly spaced. [Also, it happens in fringe projections when one projects colored fringe patterns [15].] To eliminate this inconvenience, generalized methods dealing with arbitrary (unevenly spaced) phase-steps have been developed [16-17].

Since in principle there are infinite possible phase-retrieval algorithms that can be generated, it is important to study their sensitivity to phase-step errors and noise. The purpose of the present work is threefold: Firstly we will present a new method for generating general phase-shifting algorithms with M arbitrarily spaced phase-steps, as generalization of the method previously proposed by the authors [17]. Secondly, we will study the conditions for which the phase-retrieval error due to phase-shift miscalibration can be minimized, and as result we present some novel error-insensitive algorithms. Thirdly, we will study the phase retrieval from interferograms with additive Gaussian noise, and deduce the condition to be satisfied for minimizing the phase-retrieval error.

The plan of the paper is as follows: In Sect. 2 we will present the generalized procedure for generating M-frame arbitrarily-spaced phase-shifting algorithms. In Sect. 3 the conditions for generating algorithms insensitive to linear phase-step errors are studied. In Sect. 4, the performance of new algorithms for phase extraction from interferograms with additive random noise, is analyzed and simulations are presented.

2 Phase-shifting algorithm with arbitrary phase-steps

Let us consider that the coefficients a_m and b_m appearing in (2) are the real and imaginary part of a complex number c_m, i.e. $c_m = a_m + ib_m$. Then, the expression (2) can be rewritten as

$$\phi = arg \sum_{m=1}^{M} c_m I_m \ , \tag{3}$$

where *arg* denotes the *argument-function.*

Using (1) one obtains

$$\sum_{m=1}^{M} c_m I_m = I_0 \sum_{m=1}^{M} c_m + \frac{I_0 K}{2} e^{i\phi} \sum_{m=1}^{M} c_m e^{i\delta_m} + \frac{I_0 K}{2} e^{-i\phi} \sum_{m=1}^{M} c_m e^{-i\delta_m}. \tag{4}$$

Hence it is clear that, given a set of values δ_m, to secure the validity of (3) the conditions to be verified by the coefficients c_m are

$$\sum_{m=1}^{M} c_m = 0 \ , \tag{5a}$$

$$\sum_{m=1}^{M} c_m \, e^{-i\delta_m} = 0, \tag{5b}$$

and

$$\sum_{m=1}^{M} c_m e^{i\delta_m} = \alpha, \tag{5c}$$

with α being an arbitrary positive real number. [It can be easily demonstrated that if the condition (5c) is not satisfied, the retrieved phase will differ from the actual one in a global constant.]

Equivalently, in terms of a_m and b_m the conditions (5a-c) can be rewritten as

$$\sum_{m=1}^{M} a_m = 0; \sum_{m=1}^{M} b_m = 0 \ ; \tag{6a}$$

$$\sum_{m=1}^{M} a_m \cos\delta_m + b_m \sin\delta_m = 0; \sum_{m=1}^{M} b_m \cos\delta_m - a_m \sin\delta_m = 0; \qquad (6b)$$

$$\sum_{m=1}^{M} b_m \cos\delta_m + a_m \sin\delta_m = 0; \sum_{m=1}^{M} a_m \cos\delta_m - b_m \sin\delta_m = \alpha. \qquad (6c)$$

Thus, when we are working with M phase-steps, we have to set $2M$ real coefficients (a_m, b_m) that verify the six equations (6a-c), so that we actually have $2M$-6 free coefficients. For example, in 3-frame PSI (i.e. for $M = 3$) we do not have free parameters to choose (besides a trivial factor multiplying all coefficients (a_m, b_m), which is canceled in the expression (2)), so that for a given set of values δ_m there exists a unique algorithm for phase retrieval. But for $M > 3$, for each set of values δ_m the phase-retrieval algorithm is not unique, and thus, from the free coefficients it is possible to choose an optimal set that minimizes the phase errors caused by miscalibration of the phase-steps and/or due to noise, as shown in the next.

3 Algorithms insensitive to phase-step miscalibration

3.1 General discussion

Let δ_m (with m = 1,2,…M) be the nominal values of the phase-steps, and δ'_m be the corresponding actual step values, i.e.

$$\delta'_m = \delta_m + \epsilon_m \,, \qquad (7)$$

where ϵ_m are the (unknown) deviations from the nominal values.

In practice, the interferograms experimentally acquired are associated to the actual step values. Thus, instead of (1) one has

$$I_m(x,y) = I_0(x,y)\left[1 + K(x,y)\cos\left(\phi(x,y) + \delta_m + \epsilon_m\right)\right] , \qquad (8)$$

Then, the phase retrieved using (3) -which we will denote as $\phi'(x, y)$ - will not be exactly the true phase $\phi(x, y)$, but, in the best case, it will differ from $\phi(x, y)$ in a trivial constant independent of the position. Using (8) and (5a) we get

$$\sum_{m=1}^{M} c_m I_m = \frac{I_0 K}{2} e^{i\phi} \sum_{m=1}^{M} c_m e^{i(\delta_m+\epsilon_m)} + \frac{I_0 K}{2} e^{-i\phi} \sum_{m=1}^{M} c_m e^{-i(\delta_m+\epsilon_m)}, \tag{9}$$

or equivalently,

$$\frac{2\left|\sum_{m=1}^{M} c_m I_m\right|}{I_0 K} e^{i(\phi'-\phi)} = \sum_{m=1}^{M} c_m e^{i(\delta_m+\epsilon_m)} + e^{-i2\phi} \sum_{m=1}^{M} c_m e^{-i(\delta_m+\epsilon_m)}. \tag{10}$$

If we are looking for a retrieved phase (ϕ') that differs from the actual phase (ϕ) at most in a trivial global constant, the factor multiplying $e^{-i2\phi}$ on the right-side of (10) must necessarily be zero, i.e.

$$\sum_{m=1}^{M} c_m e^{-i(\delta_m+\epsilon_m)} = 0. \tag{11}$$

This last expression can be simplified by considering that the phase-step deviations from their nominal values are small, i.e. $\epsilon_m << 1$. Then, at first order in ϵ_m one has $e^{-i\epsilon_m} \approx 1 - i\epsilon_m$. Thus, using (5b), the expression (11) reduces to

$$\sum_{m=1}^{M} c_m \epsilon_m e^{-i\delta_m} = 0 . \tag{12}$$

On the other hand, in a first-order approximation we can write the first term on the right-side of (10) as

$$\sum_{m=1}^{M} c_m e^{i(\delta_m+\epsilon_m)} \approx \sum_{m=1}^{M} c_m e^{i\delta_m} + i \sum_{m=1}^{M} c_m \epsilon_m e^{i\delta_m}. \tag{13}$$

Hence it is clear that $\phi'(x, y)$ will be independent of deviations of the phase-steps (ϵ_m) if the condition

$$\sum_{m=1}^{M} c_m \epsilon_m e^{i\delta_m} = 0 \tag{14}$$

is verified. This last condition can be important in the eventual case that the deviations ϵ_m vary over the spatial position, i.e. $\epsilon_m = \epsilon_m(x, y)$, see e.g. [5,18].

3.2 Linear phase-step error

Let as assume that we have a linear phase-step error, i.e.

$$\epsilon_m = \epsilon_0 \delta_m, \tag{15}$$

where $\epsilon_0$is a coefficient determined by the shift device. The linear phase error is often due to a calibration error, e.g. due to an erroneous voltage increment for each bucket in case of a piezoelectric controlled phase shift.

Substituting (15) into (12), it reduces to

$$\sum_{m=1}^{M} c_m \delta_m e^{-i\delta_m} = 0 \ , \tag{16}$$

And substituting (15) into (14), we get

$$\sum_{m=1}^{M} c_m \delta_m e^{i\delta_m} = 0. \tag{17}$$

Equivalently, in terms of a_m and b_m the conditions (16) and (17) can be rewritten as

$$\sum_{m=1}^{M} \delta_m [a_m \cos \delta_m + b_m \sin \delta_m] = 0 \ ; \sum_{m=1}^{M} \delta_m [b_m \cos \delta_m - a_m \sin \delta_m] = 0; \tag{18}$$

$$\sum_{m=1}^{M} \delta_m [a_m \cos \delta_m - b_m \sin \delta_m] = 0 \ ; \sum_{m=1}^{M} \delta_m [b_m \cos \delta_m + a_m \sin \delta_m] = 0. \tag{19}$$

Summarizing, expressions (6a-c) are the six basic conditions to be verified in order to retrieve the phase $\phi(x, y)$ using the algorithm shown in (2). The two additional conditions shown in (18) are basic conditions to recover, in a first-order approximation, a phase distribution $\phi'(x, y)$ that differs from the actual phase $\phi(x, y)$ at most in a trivial global constant in presence of linear phase-step error. The expressions (19) are the additional conditions to be verified to recover a phase distribution independent of the deviation ϵ_0. [These last two conditions are nor essential, and are important only in case that $\epsilon_0 = \epsilon_0(x,y)$.] Thus, the coefficients a_m and b_m must satisfy at least eight conditions ((6a-c) and (18)) to minimize (in a first order approximation) the phase-retrieval error induced by the linear phase-step error. In other words, for a given set of phase-steps, one can choose $2M-8$ free coefficients (a_m and b_m) for generating different algorithms insensitive to linear phase-step error.

3.3 Particular cases of 4-frame algorithms

For the purpose to illustrate the generation of phase retrieval algorithms insensitive to the linear phase-step error, let us consider the particular case with $M = 4$. In such case, there are not free coefficients to choose, and thus, for each given set of phase-steps δ_m there is only a unique algorithm of the form shown in (2) that is insensitive to the linear phase-step error.

3.3.1 Example of four evenly spaced phase-steps with $\delta_m = (m-1)2\pi/3$

In this particular case the coefficients that satisfy the conditions (6a-c) and (18) are $a_1 = \sqrt{3}$, $a_2 = -\sqrt{3}$, $a_3 = -\sqrt{3}$, $a_4 = \sqrt{3}$, $b_1 = -1$, $b_2 = -3$, $b_3 = 3$ and $b_4 = 1$, so that the phase-retrieval algorithm can be written as

$$\tan(\phi) = \frac{1}{\sqrt{3}} \frac{-I_1 - 3I_2 + 3I_3 + I_4}{I_1 - I_2 - I_3 + I_4}.$$

And thus, the proposed approach recover, for this particular case, the algorithm originally proposed by Surrel [1,6].

3.3.2 Example of four evenly spaced phase-steps with $\delta_m = (m-1)\pi/2$

In this particular case the coefficients that satisfy the conditions (6a-c) and (18) are $a_1 = 1$, $a_2 = 1$, $a_3 = -3$, $a_4 = 1$, $b_1 = 1$, $b_2 = -3$, $b_3 = 1$ and $b_4 = 1$, so that the phase-retrieval algorithm is

$$\tan(\phi) = \frac{I_1 - 3I_2 + I_3 + I_4}{I_1 + I_2 - 3I_3 + I_4},$$

which is a well-known algorithm proposed by Schwider et al. [1,7].

3.3.3 Example of four unevenly spaced phase-steps

As example of application of the proposed procedure, we now consider the case of four (arbitrarily chosen) unevenly spaced phase-steps. We arbitrarily choose δ_1 = 0, δ_2 = π/4, δ_3= 5π/3 and δ_4 = 7π/2 . For this particular set of δ_m-values, the coefficients that satisfy simultaneously the conditions (6a-c) and (18) are (besides a trivial factor)

$$a_1 = 1.0231...;\ a_2 = -0.5818...;\ a_3 = -2.2235...;\ a_4 = -0.2179...; \tag{20a}$$

$$b_1 = -05511...,\ b_2 = -0.0636...,\ b_3 = 1.1922... \text{ and } b_4 = -0.5774.... \tag{20b}$$

Substituting these values in (2), one gets an algorithm insensitive to the linear phase-step error (valid for the particular set of phase-steps given above).

The performance of the algorithm depends on the precise values for the coefficients. Figure 1 shows the results of a simulation in which we calculate the MSE of the retrieved phase as function of the deviation (ϵ_0) for different (non-optimal) values of a_1 . The solid line depicts the MSE evolution for the optimal values showed in (20a-b), while the dashed curve A corresponds to $a_1 = 1.1\times 1.0231$ and the curve B corresponds to $a_1 = 0.9\times 1.0231$.[When the value of a_1 is modified, the other coefficients a_m (with $m \neq 1$) and b_m are recalculated in order to verify the conditions (6a-c).]

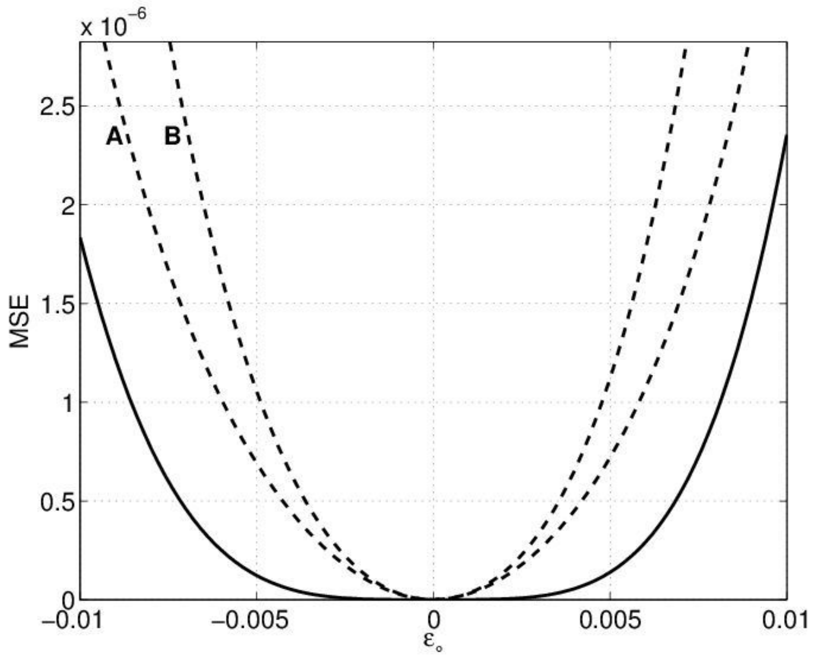

Fig. 1. Variations of the MSE as function of ϵ_0 for different values of the coefficient a_1 ; the continuous curve corresponds to the optimal value $a_1 = 1.0231$, while dashed curve A corresponds to $a_1 = 1.1 \times 1.0231$ and curve B corresponds to $a_1 = 0.9 \times 1.0231$.

4 Phase retrieval from interferograms with additive random noise

4.1 Estimation of the noise-induced phase error

In the next we will model a noisy interferogram ($I'_m(x,y)$) as the addition of an ideal noise-free interferogram ($I_m(x,y)$) plus white Gaussian noise ($n_m(x,y)$), i.e.

$$I'_m(x,y) = I_m(x,y) + n_m(x,y). \tag{21}$$

By defining

$$S' = \sum_{m=1}^{M} c_m I'_m(x,y); \text{ and } S = \sum_{m=1}^{M} c_m I_m , \tag{22}$$

one gets

$$S' = S + \sum_{m=1}^{M} c_m n_m . \tag{23}$$

From (3) and (22) it is clear that the actual phase is given by $\phi = arg(S)$, while the phase obtained from the noisy interferograms is given by $\phi' = arg(S')$. Also, from (4) and (5a-c), it is not difficult to demonstrate that

$$|S| = |\sum_{m=1}^{M} c_m I_m| = \frac{I_0 K}{2}\left|\sum_{m=1}^{M} c_m e^{i\delta_m}\right| = I_0 K\alpha/2 \,. \tag{24}$$

If we define the noise as a complex vector $N = \sum_{m=1}^{M} c_m n_m$, we can rewrite (23) as

$$S' = S + N. \tag{25}$$

Figure 2 shows a phasor representation of (25) on the complex plane; the circle represents the points where it is more probable to find S'. By calculating the radius of the circle is possible to obtain the maximum deviation of the calculated phase with respect to its actual value.

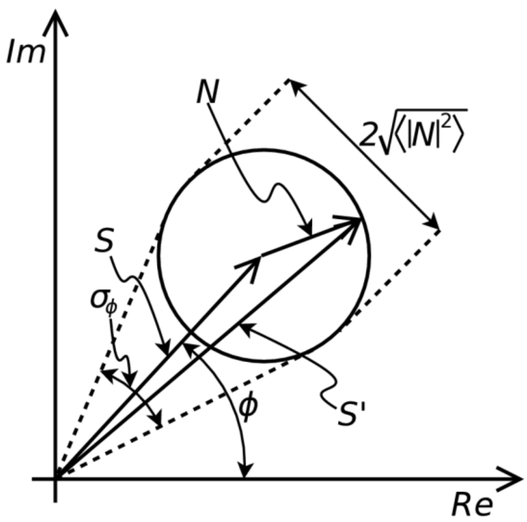

Fig. 2. Phasor representation of the expression (25). $\sigma_\phi/2$ is the maximum deviation of the retrieved phase (ϕ') with respect to the actual phase (ϕ).

We will estimate the circle radius as square root of the expected value of $|N|^2$. The expected value of $|N|^2$ is given by (see Appendix A for a deduction)

$$< |N|^2 > = \sigma_n^2 \sum_{m=1}^{M}(a_m^2 + b_m^2), \tag{26}$$

where <…> denotes *expected value* and σ_n is the standard deviation of the noise.

Let us denote as $\sigma_\phi/2$ the maximum deviation of the retrieved phase (ϕ') with respect to the actual phase (ϕ). From Fig, 2 it is clear that $\sin(\frac{\sigma_\phi}{2}) = \frac{\sqrt{<|N|^2>}}{|S|}$, and thus, if σ_ϕ is small one gets

$$\sigma_\phi^2 \approx 4 < |N|^2 > /|S|^2. \tag{27}$$

Now, substituting (24) and (26) into (27), it results

$$\sigma_\phi^2 \approx 16\sigma_n^2/(KI_0\alpha)^2 \sum_{m=1}^{M}(a_m^2 + b_m^2). \tag{28}$$

As expected, σ_ϕ depends on the relationship between the noise standard deviation and the local contrast (KI_0) of the images. Also, it depends on the utilized algorithm, so that by choosing properly the coefficients a_m and b_m it is possible to minimize the noise-induced phase error σ_ϕ, as will be discussed in the next.

4.2 Minimization of the phase-retrieval error induced by noise

For a given set of phase-steps δ_m, in order to minimize the noise-induced phase error, σ_ϕ, we have to choose properly the parameters a_m and b_m to minimize the expression

$$f(a_1, \ldots, a_M, b_1, \ldots, b_M) = (1/2) \sum_{m=1}^{M}(a_m^2 + b_m^2). \tag{29}$$

[The factor α in (28) will be canceled after the mentioned minimization, as shown below.]

The target function given by Eq. (29) must be minimized subject to Eqs. (6a-c), which can be done, for instance, using Lagrange's multipliers and defining

$$g_1(a_1, \ldots, a_M, b_1, \ldots, b_M) = \sum_{m=1}^{M} a_m\ , \tag{30a}$$

$$g_2(a_1, \ldots, a_M, b_1, \ldots, b_M) = \sum_{m=1}^{M} b_m\ , \tag{30b}$$

$$g_3(a_1, \ldots, a_M, b_1, \ldots, b_M) = \sum_{m=1}^{M}(a_m \cos\delta_m + b_m \sin\delta_m)\ , \tag{30c}$$

$$g_4(a_1, \ldots, a_M, b_1, \ldots, b_M) = \sum_{m=1}^{M}(b_m \cos\delta_m - a_m \sin\delta_m)\ , \tag{30d}$$

$$g_5(a_1,\dots,a_M,b_1,\dots,b_M) = \sum_{m=1}^{M}(a_m\cos\delta_m - b_m\sin\delta_m) - \alpha\ , \quad (30e)$$

$$g_6(a_1,\dots,a_M,b_1,\dots,b_M) = \sum_{m=1}^{M}(b_m\cos\delta_m + a_m\sin\delta_m)\ , \quad (30f)$$

the constraints are

$$g_k(a_1,\dots,a_M,b_1,\dots,b_M) = 0\ , \quad (31)$$

with $k = 1, 2..., 6$.

Applying the Lagrange multipliers method we have to solve the set of equations

$$\frac{\partial f}{\partial a_m} = \sum_{k=1}^{6} \lambda_k \frac{\partial g_k}{\partial a_m}\ ; \frac{\partial f}{\partial b_m} = \sum_{k=1}^{6} \lambda_k \frac{\partial g_k}{\partial b_m}\quad , \quad (32)$$

for$m$ = 1,2,..., M.

So, from (29)-(32) we have

$$a_m = \lambda_1 + \lambda_3\cos\delta_m - \lambda_4\sin\delta_m + \lambda_5\cos\delta_m + \lambda_6\sin\delta_m \quad (33)$$

and

$$b_m = \lambda_2 + \lambda_3\sin\delta_m + \lambda_4\cos\delta_m - \lambda_5\sin\delta_m + \lambda_6\cos\delta_m. \quad (34)$$

Substituting (33) and (34) into (31) we have a 6 × 6 linear system to calculate the Lagrange multipliers. The solutions of the system are

$$\lambda_1 = \alpha\frac{K_1M - K_2K_4 - K_1K_3}{-2K_1^2K_3 + 2K_1^2M - 4K_1K_2K_4 + 2K_2^2K_3 + 2K_2^2M + K_3^2M + K_4^2M - M^3}, (35)$$

$$\lambda_2 = \alpha\frac{K_1K_4 - K_2K_3 - K_2M}{-2K_1^2K_3 + 2K_1^2M - 4K_1K_2K_4 + 2K_2^2K_3 + 2K_2^2M + K_3^2M + K_4^2M - M^3}, (36)$$

$$\lambda_3 = \alpha\frac{-K_1^2 + K_2^2 + K_3M}{-2K_1^2K_3 + 2K_1^2M - 4K_1K_2K_4 + 2K_2^2K_3 + 2K_2^2M + K_3^2M + K_4^2M - M^3}, (37)$$

$$\lambda_4 = \alpha\frac{2K_1K_2 - K_4M}{-2K_1^2K_3 + 2K_1^2M - 4K_1K_2K_4 + 2K_2^2K_3 + 2K_2^2M + K_3^2M + K_4^2M - M^3}, (38)$$

$$\lambda_5 = \alpha\frac{K_1^2 + K_2^2 - M^2}{-2K_1^2K_3 + 2K_1^2M - 4K_1K_2K_4 + 2K_2^2K_3 + 2K_2^2M + K_3^2M + K_4^2M - M^3}, (39)$$

$$\lambda_6 = 0, \tag{40}$$

where

$$K_1 = \sum_{m=1}^{M} \cos\delta_m , \tag{41}$$

$$K_2 = \sum_{m=1}^{M} \sin\delta_m , \tag{42}$$

$$K_3 = \sum_{m=1}^{M} (\cos^2\delta_m - \sin^2\delta_m) , \tag{43}$$

$$K_4 = 2\sum_{m=1}^{M} \cos\delta_m \sin\delta_m . \tag{44}$$

Thus, given a set of values δ_m, from expressions (41)-(44)we can calculate K_{1-4} and then from (35)-(40) one obtains λ_{1-6}. Then, substituting these values in (33)-(34) we calculate the optimal parameters set a_m and b_m that minimize the phase-retrieval error.

Note that the Lagrange multiplier $\lambda_6 = 0$ (Eq. 40) implies that the coefficients that minimize the noise propagation are the same independently of the fact that the retrieved phase could differ from the actual phase in a global constant. In other words, the first Eq. (6c) does not play a role in the minimization process.

For example, if we apply the described procedure for the set $\delta_m = (m - 1)\, 2\pi/3$ (with *m*= 1,2,3,4), one easily obtains $a_1 = 1$, $a_2 = -1$, $a_3 = -1$, $a_4 = 1$, and $b_1 = 0$, $b_2 = -\sqrt{3}$, $b_3 = \sqrt{3}$, $b_4 = 0$, so that the phase-retrieval algorithm is

$$\tan(\phi) = \sqrt{3}\frac{I_3 - I_2}{I_1 - I_2 - I_3 + I_4},$$

and thus, we recover for this particular case the algorithm proposed by Larkin and Oreb [1,8]. Now, if we apply the error minimization procedure for the set $\delta_m = (m - 1)\, \pi/2$ (with *m*= 1,2,3,4), one obtains $a_1 = 1, a_2 = 0$, $a_3 = -1$, $a_4 = 0$, and $b_1 = 0$, $b_2 = -1$, $b_3 = 0, b_4 = 1$, so that the phase-retrieval algorithm will be

$$\tan(\phi) = \frac{I_4 - I_2}{I_1 - I_3},$$

which is the well-known 4-frame algorithm proposed by Bruning et al. [1,9].

In order to illustrate the proposed error-minimization procedure with unevenly spaced phase-steps, let us consider an arbitrary set of phase-steps, for example $\delta_1 = 0$, $\delta_2 = \pi/4$, $\delta_3 = 5\pi/3$ and $\delta_4 = 7\pi/2$. In this case, the coefficients that minimize the deviation are (besides a factor)

$$a_1 = 0.6287\,, a_2 = -0.3915\,,\ a_3 = 0.2964\,,\ a_4 = -0.5335\,, \tag{45}$$

and

$$b_1 = 0.1563\,,\ b_2 = -0.4049\,,\ b_3 = 0.2599\,,\ \ b_4 = -0.0114\,. \tag{46}$$

To illustrate the performance of the generated algorithm, we utilized as test phase the function shown in Fig. 3.

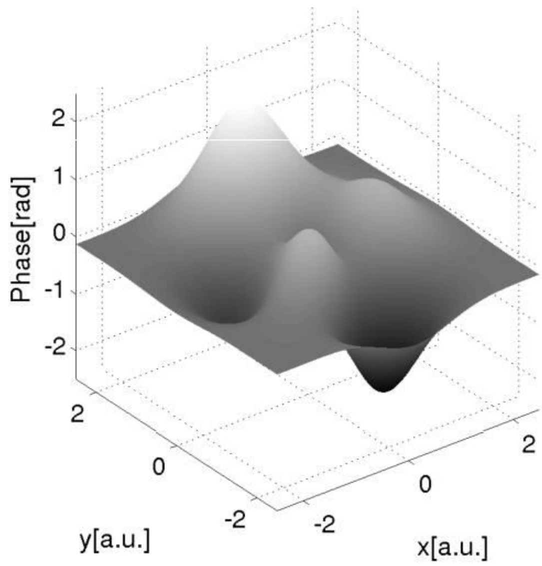

Fig. 3. Test phase utilized for illustrating the performance of the generated phase-retrieval algorithm. (Image size 501x501 pixel.)

Utilizing this test phase, we synthesized four interferograms with phase-steps $\delta_1 = 0$, $\delta_2 = \pi/4$, $\delta_3 = 5\pi/3$ and $\delta_4 = 7\pi/2$, and added to them certain (arbitrary) amount of Gaussian noise.

Figures 4(a)-(e) are excerpts of a video sequence (see Visualization 1) that shows the retrieved phase for different sets of a_m and b_m parameters as shown in Table 1. Of course, all these parameter sets verify the Eqs. 6(a)-(c), so that they generate valid four-frame phase-retrieval algorithms, but they do not have their optimal values for minimizing the noise, with the exception of the phase shown in Fig. 4(e) that was retrieved using the optimal parameter values.

Clearly, for the optimal values of a_m and b_m the noise in the retrieved phase is minimal as demonstrated by comparing Fig. 4(e) with the original phase shown in Fig. 3.

Table 1. Sets of values a_m and b_m for generating phase-retrieval algorithms whose performance is illustrated in Figs. 4(a)-(e).

Fig. 4	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
(a)	4.5442	-2.2802	-4.8637	2.5997	0.7481	-0.6903	-0.5199	0.4621
(b)	1.1853	0.4835	2.6869	-1.9851	2.5457	-1.5574	-2.8889	1.9006
(c)	1.6292	-0.8742	-1.0222	0.2671	-1.5754	0.4305	2.5421	-1.3971
(d)	0.0356	-0.1054	1.0780	-1.0081	-0.6488	-0.0165	1.3210	-0.6556
(e)	0.6287	-0.3915	0.2963	-0.5335	0.1563	-0.4049	0.2599	-0.0114

The video sequence (Visualization 1) shows the evolution of the retrieved (noisy) phase when the parameters a_1 and b_1 approximate in a spiral way their optimal values show in (45) and (46). For each pair of values (a_1, b_1) the other six parameters $a_{2,3,4}$ and $b_{2,3,4}$ are recalculated to verify (6a-c)), as shown in Table 1. [Note that the video sequence is not showing an iterative algorithm, but simply it is illustrating the performance of algorithms obtained with different values of the coefficients a_1 and b_1.]

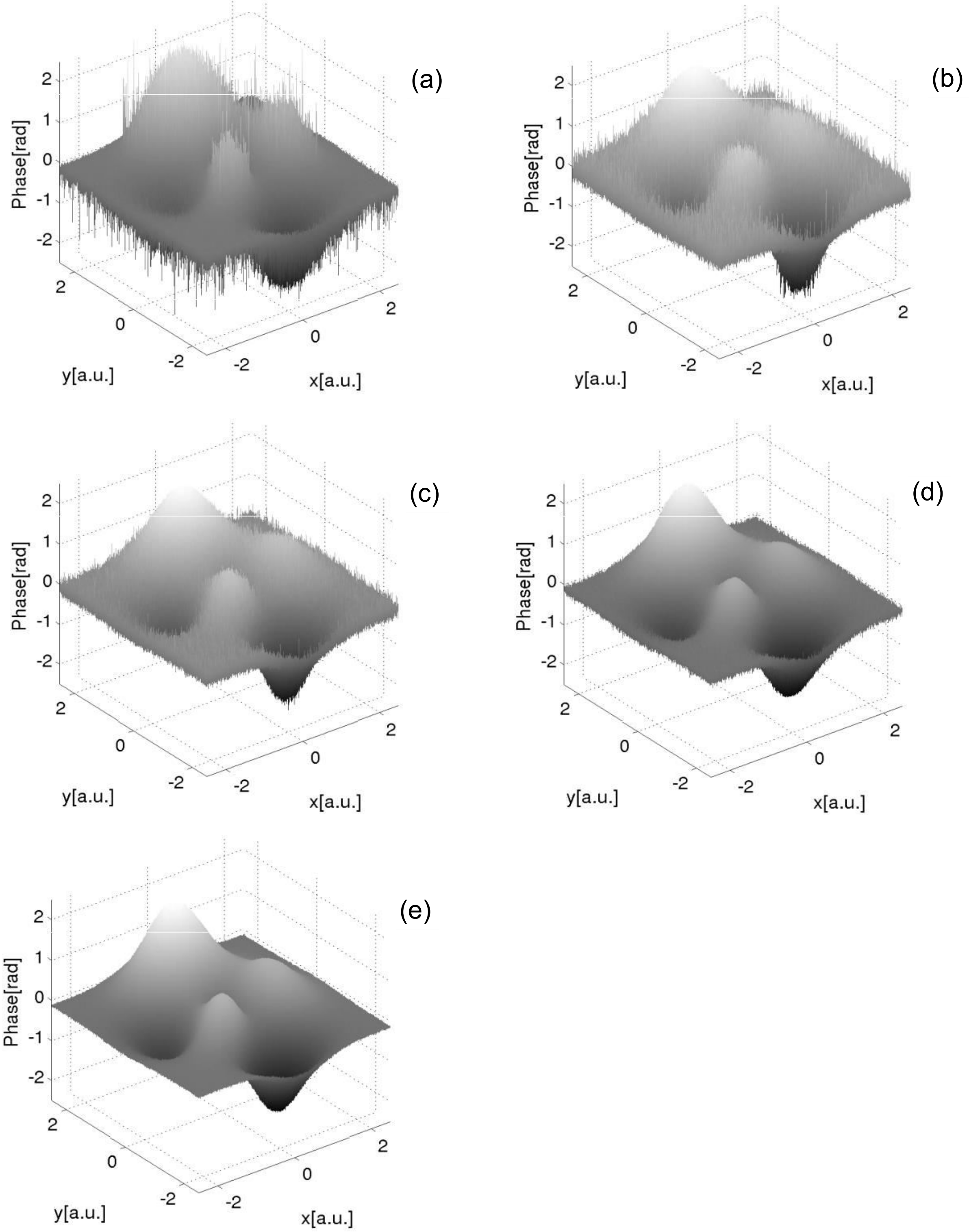

Fig. 4(a)-(e). Retrieved phase for different sets of parameter values shown in Table 1.

Figure 5 shows the mean square error (MSE) of the retrieved phase as function of the values of a_1 and b_1, with a minimum at the calculated optimal values.

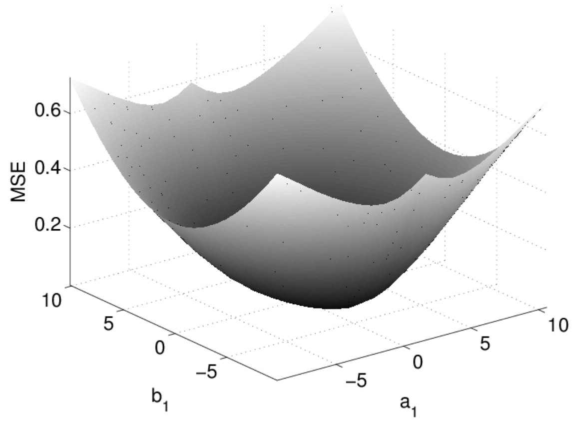

Fig. 5. MSE of the retrieved phase as function of a_1 and b_1 values.

5 Conclusions

We have presented a new method for generating general phase-shifting algorithms for a given set of M arbitrarily spaced phase-steps (δ_m). Specifically, we demonstrated that for generating a phase-retrieval algorithm we have to set $2M$ real coefficients (a_m, b_m) that verify the six equations (6a-c). Thus, for $M > 3$ we actually have $2M$-6 free coefficients, so that for each set δ_m we will able to generate a multiplicity of phase-retrieval algorithms. In particular, if one chooses the free parameters so that they verify the two additional conditions shown in (18), in presence of linear phase-step errors (e.g., due to miscalibration) one recovers in a first-order approximation a phase distribution that differs from the actual phase at most in a trivial global constant. While if the two additional conditions shown in (19) are verified, one recovers a phase distribution independent of the deviation ϵ_0.

Also we have considered the phase retrieval from interferograms with additive random (Gaussian) noise. Applying the method of Lagrange multipliers we have calculated the set of parameters (a_m, b_m) that minimizes the phase-retrieval error due to the noise for given set of phase-steps. As illustration of the error minimization procedure, we presented a simulation of the MSE obtained by changing the values of the parametersa_m and b_m around their optimal values.

Appendix A

Let us define the complex noise vector $N = \sum_{m=1}^{M} c_m n_m$, with n_m being random real numbers and c_m complex. In the next we will write $c_m = a_m + ib_m$.

Then the module of $N$is

$$|N|^2 = (\sum_{m=1}^{M} a_m n_m)^2 + (\sum_{m=1}^{M} b_m n_m)^2, \quad \text{(A.1)}$$

so its expected value is

$$< |N|^2 >=< (\sum_{m=1}^{M} a_m n_m)^2 > +< (\sum_{m=1}^{M} b_m n_m)^2 >. \quad \text{(A.2)}$$

On the other hand, the standard deviation (σ_a) of $\sum_{m=1}^{M} a_m n_m$ verifies

$$\sigma_a^2 =< (\sum_{m=1}^{M} a_m n_m)^2 > -< (\sum_{m=1}^{M} a_m n_m) >^2. \quad \text{(A.3)}$$

Since $< (\sum_{m=1}^{M} a_m n_m) >= \sum_{m=1}^{M} a_m < n_m >$, if we assume that the additive noise have null expected value, it results $< (\sum_{m=1}^{M} a_m n_m) >= 0$, and thus from (A.3) one obtains

$$\sigma_a^2 =< (\sum_{m=1}^{M} a_m n_m)^2 >. \quad \text{(A.4)}$$

On the other hand, since σ_a is the standard deviation of $\sum_{m=1}^{M} a_m n_m$ and we assume the noise have normal distribution, then we will have

$$\sigma_a^2 = \sum_{m=1}^{M} a_m^2 \sigma_m^2, \quad \text{(A.5)}$$

where $\sigma_m$is the standard deviation of the noise (n_m) present in the m-th interferogram.

It is reasonable to assume that all images (interferograms) have the same noise distribution, so that $\sigma_m = \sigma_n$(constant) for $m = 1 \dots M$, and then from (A.5) we get

$$\sigma_a^2 = \sigma_n^2 \sum_{m=1}^{M} a_m^2 \qquad \text{(A.6)}$$

Then, from (A.4) and (A.6) it results

$$< (\sum_{m=1}^{M} a_m n_m)^2 >= \sigma_n^2 \sum_{m=1}^{M} a_m^2. \qquad \text{(A.7)}$$

In a similar way, we obtain

$$< (\sum_{m=1}^{M} b_m n_m)^2 >= \sigma_n^2 \sum_{m=1}^{M} b_m^2. \qquad \text{(A.8)}$$

Now, substituting (A.7) and (A.8) into (A.2), it finally results

$$< |N|^2 > = \sigma_n^2 \sum_{m=1}^{M} (a_m^2 + b_m^2), \qquad \text{(A.9)}$$

or equivalently,

$$< |N|^2 > = \sigma_n^2 \sum_{m=1}^{M} c_m^2. \qquad \text{(A.10)}$$

AppendixB

In this appendix we will discuss an alternative approach to generate phase-retrieval algorithms taking into account the conditions shown in Eqs. 6(a)-(c).

Let γ_{ml} (with$m, l = 1,2, ..M$) be an antisymmetric matrix, i.e.$\gamma_{ml} = -\gamma_{lm}$, that verifies the condition

$$\sum_{m=1}^{M} \gamma_{ml} = 0 . \qquad \text{(B.1)}$$

Then, we define $c_m$as

$$c_m = i \sum_{l=1}^{M} \gamma_{ml} e^{-i\delta_l}, \qquad \text{(B.2)}$$

and

$$S = i \sum_{l=1}^{M} (\sum_{m=1}^{M} I_m \gamma_{ml} e^{-i\delta_l}). \qquad \text{(B.3)}$$

We have to demonstrate that $c_m$defined in equation (B.2) verifies the conditions 5(a)-(c).

From (B.1) and (B.2), we have

$$\sum_{m=1}^{M} c_m = \sum_{l=1}^{M} i\, e^{-i\delta_l} \sum_{m=1}^{M} \gamma_{ml} = 0, \qquad \text{(B.3)}$$

which is the condition 5(a).

From (B.2) and the antisymmetry of γ_{ml}, one obtains

$$\sum_{m=1}^{M} c_m\, e^{-i\delta_m} = i \sum_{l=1}^{M} (\sum_{m=1}^{M} \gamma_{ml}\, e^{-i(\delta_l+\delta_m)}) = 0, \qquad \text{(B.4)}$$

which is the condition 5(b).

Now we will calculate $\alpha = \sum_{m=1}^{M} c_m e^{i\delta_m}$ using (B.2)

$$\alpha = \sum_{m=1}^{M} i e^{i\delta_m} \sum_{l=1}^{M} \gamma_{ml} e^{-i\delta_l} = i \sum_{l=1}^{M} (\sum_{m=1}^{M} \gamma_{ml}\, e^{i(\delta_m-\delta_l)}), \qquad \text{(B.5)}$$

or equivalently,

$$\alpha = i \sum_{l=1}^{M} [\sum_{m=1}^{M} \gamma_{ml} (\cos(\delta_m - \delta_l) + i sin(\delta_m - \delta_l))]. \qquad \text{(B.6)}$$

Since γ_{ml} is antisymmetric and $\cos(\delta_m - \delta_l) = \cos(\delta_l - \delta_m)$, the double sum of the first term of the right-side of (B.6) is zero, so

$$\alpha = - \sum_{l=1}^{M} [\sum_{m=1}^{M} \gamma_{ml}\, sin(\delta_m - \delta_l)] \qquad \text{(B.7)}$$

is a real number, which is the condition 5(c). [In principle, α can be positive or negative, but it is possible to demonstrate that if $\phi = arg(S)$(with S defined in (22)) α should be positive, while in other case $\phi = arg(S) + \pi$.]

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