Characterization of Logarithmic Fekete Critical Configurations of at Most Six Points in All Dimensions

ISSAC - Guanajuato, México - August 1, 2025

Diego Armentano, Leandro Bentancur, Federico Carrasco, Marcelo Fiori, **Matías Valdés**, and Mauricio Velasco

UNIVERSIDAD DE LA REPÚBLICA URUGUAY

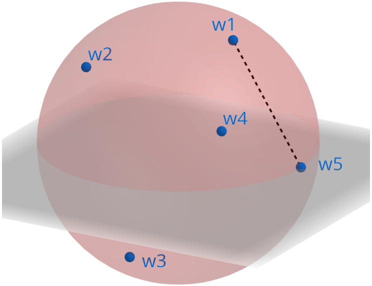
Outline

Fekete Problem

Position n points on the sphere $S^d \subset \mathbb{R}^{d+1}$, so as to maximize the product of the distances between pairs of points:

$$\prod_{1\leq i<j\leq n} \|w_i - w_j\|,$$

$$w_i \in S^d \subset \mathbb{R}^{d+1}.$$

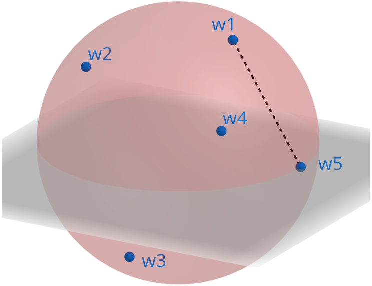


Fekete Problem

Position n points on the sphere $S^d \subset \mathbb{R}^{d+1}$, so as to maximize the product of the distances between pairs of points:

$$\prod_{1 \le i < j \le n} \|w_i - w_j\|,$$

$$w_i \in S^d \subset \mathbb{R}^{d+1}.$$



Equivalently: place the points to minimize the logarithmic energy

$$E_{\log}(w) := - \sum_{1 \le i < j \le n} \log\left(\|w_i - w_j\|^2\right), \quad w_i \in S^d \subset \mathbb{R}^{d+1}.$$

Why caring about this problem?

Fekete problem: minimize the logarithmic energy on S^d:

$$E_{\log}(w) := -\sum_{1\leq i<j\leq n} \log\left(\|w_i - w_j\|^2\right), \quad w_i \in S^d \subset \mathbb{R}^{d+1}.$$

Why caring about this problem?

Fekete problem: minimize the logarithmic energy on S^d:

$$E_{\log}(w) := -\sum_{1\le i<j\le n} \log\left(\|w_i - w_j\|^2\right), \quad w_i \in S^d \subset \mathbb{R}^{d+1}.$$

Bezout III - Shub and Smale (1993) [1]

Fekete solutions (projected to the plane) are useful for initializing homotopy algorithms. Even solutions up to a logarithmic factor.

Why caring about this problem?

Fekete problem: minimize the logarithmic energy on S^d:

$$E_{\log}(w) := -\sum_{1\leq i<j\leq n} \log\left(\|w_i - w_j\|^2\right), \quad w_i \in S^d \subset \mathbb{R}^{d+1}.$$

Bezout III - Shub and Smale (1993) [1]

Fekete solutions (projected to the plane) are useful for initializing homotopy algorithms. Even solutions up to a logarithmic factor.

Smale's 7th problem for the next Century (1998) [4]

Give an "efficient" algorithm that, for each number of points n, finds $\hat{w} \in (S^2)^n$ with optimal value up to a logarithmic factor:

$$E_{\log}(\hat{w}) - E^*_{\log} \leq c\log(n), \quad \text{for a universal constant } c.$$

What is known about Fekete problem

- Optimal configurations known for few values of n and d.
- In S^2, optimals are known only for: $n = 2, 3, 4, 5, 6$ and 12.
- Optimal value known asymptotically, up to linear term [9]:

$$\text{in } S^2: \quad E^*_{\log} = \left(\frac{1}{2} - \log 2\right)\frac{n^2}{2} - \frac{n \log n}{4} + Cn + o(n), \quad C = ?$$

- Any critical configuration satisfies: $\sum_{i=1}^{n} w_i = \vec{0}$.
- Has saddle-points [8, 10], and degenerate minima [11].
- Numerically: number of spurious local minima in S^2 increases "dramatically" with n [2]; although none is known in S^2.
- Only one spurious local minima is known [8, 10]: $n = 6$ in S^3.

Contributions and Approach

Contributions

- Formulate critical confs. as solutions of a polynomial system.
- Formulation is useful for any sphere S^d.
- For $n \leq 6$ points, and all sphere dimensions d, we:
    - recover previously known results in a Unified Framework.
    - find and classify all Critical Configurations, including unknown Saddle.

Contributions and Approach

Contributions

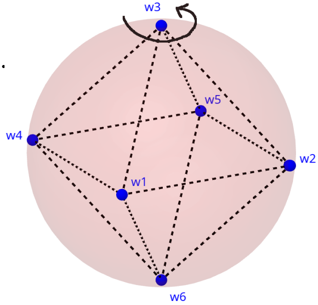
- Formulate critical confs. as solutions of a polynomial system.
- Formulation is useful for any sphere S^d.
- For $n \leq 6$ points, and all sphere dimensions d, we:
  - recover previously known results in a Unified Framework.
  - find and classify all Critical Configurations, including unknown Saddle.

Approach

- Determine the number of exact solutions m of the formulation.
- Find as many solutions as possible, until we reach m.

Removing Orthogonal Symmetry

$$E_{\log}(w) := -\sum_{1 \le i < j \le n} \log\left(\|w_i - w_j\|^2\right).$$

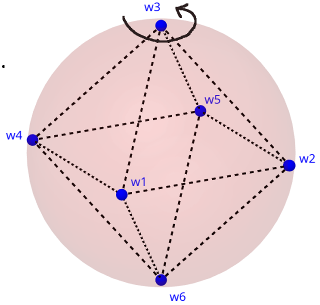


Energy is invariant under rotations of the sphere.

This gives ∞ solutions (but we need to count solutions).

Removing Orthogonal Symmetry

$$E_{\log}(w) := - \sum_{1 \le i < j \le n} \log\left(\|w_i - w_j\|^2\right).$$



Energy is invariant under rotations of the sphere.

This gives ∞ solutions (but we need to count solutions).

Change of variables

Instead of the w_i, use dot products ("angles") as variables:

$$x_{ij} := w_i^T w_j = \cos(\theta_{ij}).$$

Rotations of a solution give the exact same solution (in the x_{ij}).

Equations in terms of dot products

Lagrange optimality conditions, expressed in the $x_{ij} := w_i^T w_j$:

$$(n-1)\,x_{ki} - \sum_{j=1, j \neq i}^{n} \frac{x_{ki} - x_{kj}}{1 - x_{ij}} = 0, \;\forall\; i \neq k.$$

Equations in terms of dot products

Lagrange optimality conditions, expressed in the $x_{ij} := w_i^T w_j$:

$$(n-1)\, x_{ki} - \sum_{j=1, j\neq i}^{n} \frac{x_{ki} - x_{kj}}{1 - x_{ij}} = 0, \ \forall \ i \neq k.$$

Introduce auxiliary variables z_{ij}, and corresponding equations:

$$\boxed{z_{ij}(1 - x_{ij}) = 1.} \tag{1}$$

Equations in terms of dot products

Lagrange optimality conditions, expressed in the $x_{ij} := w_i^T w_j$:

$$(n-1)\, x_{ki} - \sum_{j=1, j\neq i}^{n} \frac{x_{ki} - x_{kj}}{1 - x_{ij}} = 0, \ \forall\ i \neq k.$$

Introduce auxiliary variables z_{ij}, and corresponding equations:

$$\boxed{z_{ij}(1 - x_{ij}) = 1.} \tag{1}$$

$$\boxed{(n-1)\, x_{ki} - \sum_{j=1, j\neq i}^{n} (x_{ki} - x_{kj}) z_{ij} = 0, \ \forall\ i \neq k.} \tag{2}$$

Gives a polynomial expression, and ensures $x_{ij} \neq 1$ $(w_i \neq w_j)$.

Equations in terms of dot products

Equations:

$$\boxed{z_{ij}(1 - x_{ij}) = 1.}$$

$$\boxed{(n-1)\, x_{ki} - \sum_{j=1, j \neq i}^{n} (x_{ki} - x_{kj}) z_{ij} = 0, \ \forall \ i \neq k.}$$

Equations in terms of dot products

Equations:

$$\boxed{z_{ij}(1 - x_{ij}) = 1.}$$

$$\boxed{(n-1)\,x_{ki} - \sum_{j=1, j\neq i}^{n} (x_{ki} - x_{kj})z_{ij} = 0, \ \forall\ i \neq k.}$$

Variables: x_{ij}, z_{ij}, $i < j$.

Equations in terms of dot products

Equations:

$$\boxed{z_{ij}(1 - x_{ij}) = 1.}$$

$$\boxed{(n-1)\,x_{ki} - \sum_{j=1, j\neq i}^{n} (x_{ki} - x_{kj})z_{ij} = 0,\ \forall\ i \neq k.}$$

Variables: $x_{ij},\ z_{ij},\ i < j.$

Number of variables and equations

	n	3	4	5	6	7	8
variables	$2\binom{n}{2}$	6	12	20	30	42	56
equations	$n(3n-1)/2$	12	22	35	51	70	92

Number of solutions for $n = 5$ points (any sphere)

Gröbner theory

- Let I be the ideal generated by the polynomials of the system.
- Given a Gröbner basis of I, its Leading Monomials determine:
  1. if the system has a finite number of solutions: $\dim(I) = 0$, and
  2. in that case, the exact number of solutions: $\deg(I)$.
- This counts solutions in $\mathbb{C}$ and counting multiplicities.

Number of solutions for $n = 5$ points (any sphere)

Gröbner theory

- Let I be the ideal generated by the polynomials of the system.
- Given a Gröbner basis of I, its Leading Monomials determine:
  1. if the system has a finite number of solutions: $\dim(I) = 0$, and
  2. in that case, the exact number of solutions: $\deg(I)$.
- This counts solutions in $\mathbb{C}$ and counting multiplicities.

For $n = 5$

$\dim(I) = 0$ and $\deg(I) = 38$ solutions.

Number of solutions for $n = 5$ points (any sphere)

Gröbner theory

- Let I be the ideal generated by the polynomials of the system.
- Given a Gröbner basis of I, its Leading Monomials determine:
  1. if the system has a finite number of solutions: $\dim(I) = 0$, and
  2. in that case, the exact number of solutions: $\deg(I)$.
- This counts solutions in $\mathbb{C}$ and counting multiplicities.

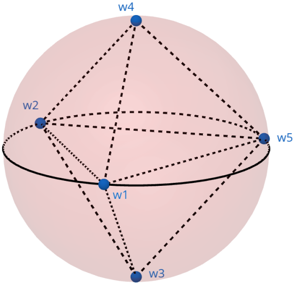
For $n = 5$

$\dim(I) = 0$ and $\deg(I) = 38$ solutions.

Next step

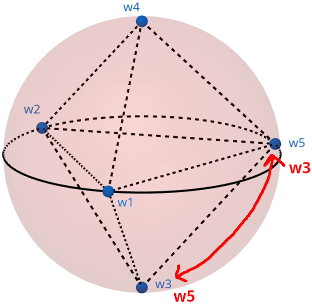
Search for solutions, until we match the 38 existent.

Finding solutions for $n = 5$ points (any sphere)



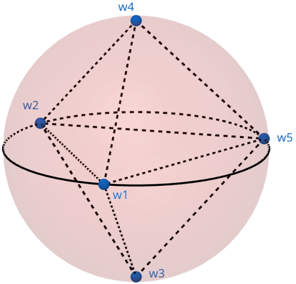
Easy to check that a particular critical conf. for $n = 5$ in S^2 is 1:3:1.

Some permutations give different solutions in x_{ij}, but of same "kind".

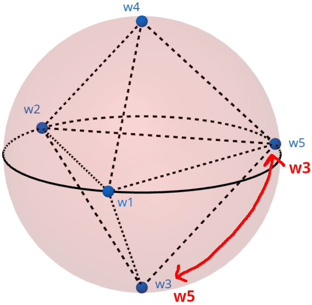


Finding solutions for $n = 5$ points (any sphere)

Easy to check that a particular critical conf. for $n = 5$ in S^2 is 1:3:1.



Some permutations give different solutions in x_{ij}, but of same "kind".

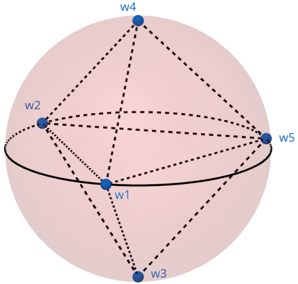


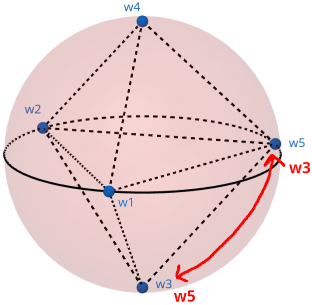
Given a "kind" of solution: how many permutations of the points w_i give different solutions in the variables x_{ij}?

Finding solutions for $n = 5$ points (any sphere)

Easy to check that a particular critical conf. for $n = 5$ in S^2 is 1:3:1.

Some permutations give different solutions in x_{ij}, but of same "kind".



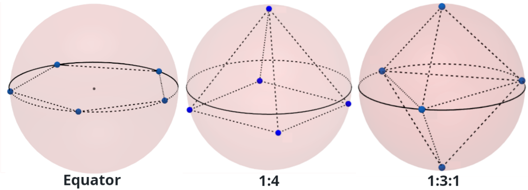


Given a "kind" of solution: how many permutations of the points w_i give different solutions in the variables x_{ij}?

We find this number by trying all possible $n!$ permutations.

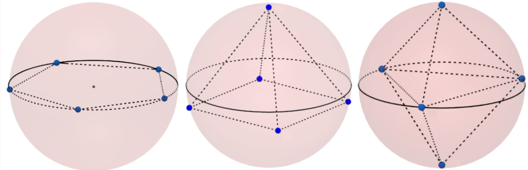
Finding solutions for $n = 5$ points (any sphere)

We can easily imagine other kind of solutions.



Finding solutions for $n = 5$ points (any sphere)

We can easily imagine other kind of solutions.



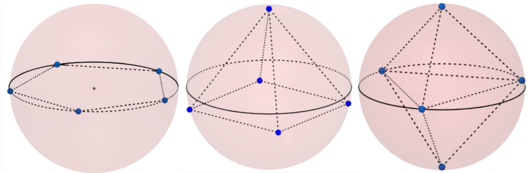
Equator 1:4 1:3:1

The different permutations of each kind are:

Conf.	Equator	1:4	1:3:1	4-simplex	Found	Existent
$\neq$ perms.	12	15	10	1	38	38
sphere	S^1	S^2		S^3		

Finding solutions for $n = 5$ points (any sphere)

We can easily imagine other kind of solutions.



The different permutations of each kind are:

Conf.	Equator	1:4	1:3:1	4-simplex	Found	Existent
$\neq$ perms.	12	15	10	1	38	38
sphere	S^1	S^2		S^3		

Theorem (For $n = 5$ points)

The only kind of critical configurations are the ones we imagined.

Number of solutions for $n = 6$ (any sphere)

We find a Gröbner base with GRevLex monomial order (*msolve*), and then use it to determine the number of solutions (*Macaulay2*).

n	$\dim I$	$\deg I$
6	0	938

Number of solutions for $n = 6$ (any sphere)

We find a Gröbner base with GRevLex monomial order (*msolve*), and then use it to determine the number of solutions (*Macaulay2*).

n	$\dim I$	$\deg I$
6	0	938

Next step: search for solutions until we match the existent 938.

Possible values of dot product for $n = 6$

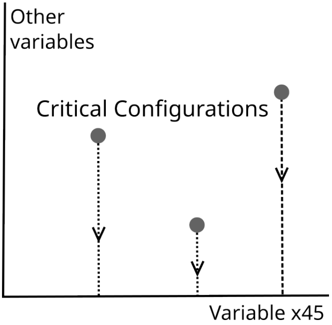
"Imaginable" solutions are not enough: as we will see, there are complex solutions.

Possible values of dot product for $n = 6$

"Imaginable" solutions are not enough: as we will see, there are complex solutions.

Approach to determine all the solutions

- Project solution set onto variable x_{45}.
- Gives possible values of x_{45}.
- Finding a Gröbner base of the ideal, with elimination order that eliminates all variables except x_{45}.



Possible values of dot product for $n = 6$

Projected ideal is a PID:

$I \cap \mathbb{Q}[x_{45}] = (p)$, some p.

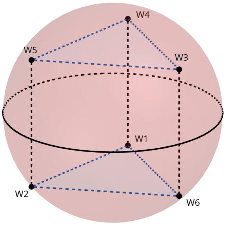
Roots of generator p are the possible values of x_{45}.

Generator Factor	Roots
x_{45}	0
$(x_{45}+1)^2$	-1
$2x_{45}-1$	1/2
$2x_{45}+1$	-1/2
$(5x_{45}-1)^2$	1/5
$5x_{45}+1$	-1/5
$(5x_{45}+7)^2$	-7/5
$(5x_{45}^2+1)^2$	$\pm\frac{i}{\sqrt{5}}$
$5x_{45}^2-22x_{45}+5$	$\frac{11\pm4\sqrt{6}}{5}$
$5x_{45}^2+2x_{45}-1$	$\frac{-1\pm\sqrt{6}}{5}$
$5x_{45}^2+14x_{45}-1$	$\frac{-7\pm3\sqrt{6}}{5}$
$25x_{45}^2+28x_{45}+19$	$\frac{-14\pm i3\sqrt{31}}{25}$
$125x_{45}^2+50x_{45}-31$	$\frac{-5\pm6\sqrt{5}}{25}$
$100x_{45}^4+95x_{45}^3-21x_{45}^2-22x_{45}+10$	4 complex
$250x_{45}^4+110x_{45}^3-21x_{45}^2-19x_{45}+4$	4 complex
$400x_{45}^4+488x_{45}^3-111x_{45}^2-196x_{45}+67$	4 complex
$3x_{45}+1$	-1/3
$5x_{45}+4$	-4/5
$10x_{45}-1$	1/10
$25x_{45}-1$	1/25
$25x_{45}+11$	-11/25
$25x_{45}+23$	-23/25

Solutions for each coordinate value of x_{45}

For each factor p_i of the generator (each x_{45} value)

1. Add $p_i = 0$ to the original system.
2. Find solutions of simplified system, and count permutations.

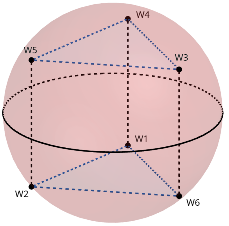


Configuration	# perms	Sphere
Equator	60	S^1
1:5	72	S^2
1:4:1	15	S^2
3:3	60+60	S^2
Complex 1	90	S^2
Complex 2	360	S^2
Real 1	15	S^3
Real 2	45	S^3
Real 3	60	S^3
Real 4	10	S^3
5-simplex	1	S^4
Found	848	
Existent	938	
Difference	938 - 848 = 90	

Solutions for each coordinate value of x_{45}

For each factor p_i of the generator (each x_{45} value)

1. Add $p_i = 0$ to the original system.
2. Find solutions of simplified system, and count permutations.



Configuration	# perms	Sphere
Equator	60	S^1
1:5	72	S^2
1:4:1	15	S^2
3:3	60+60	S^2
Complex 1	90	S^2
Complex 2	360	S^2
Real 1	15	S^3
Real 2	45	S^3
Real 3	60	S^3
Real 4	10	S^3
5-simplex	1	S^4
Found	848	
Existent	938	
Difference	938 - 848 = 90	

Complex 1 has multiplicity 2: +90.

Classification of **real** critical configurations ($n = 6$)

Configuration	S^1	S^2	S^3	S^4
Equator	GM	S	S	S
1:5	-	S	S	S
1:4:1	-	GM	S	S
3:3 ($\sqrt{6}$)	-	S	S	S
Real 1	-	-	SM	S
Real 2	-	-	S	S
Real 3	-	-	S	S
Real 4	-	-	GM	S
5-simplex	-	-	-	GM

Classification of **real** critical configurations ($n = 6$)

Configuration	S^1	S^2	S^3	S^4
Equator	GM	S	S	S
1:5	-	S	S	S
1:4:1	-	GM	S	S
3:3 ($\sqrt{6}$)	-	S	S	S
Real 1	-	-	SM	S
Real 2	-	-	S	S
Real 3	-	-	S	S
Real 4	-	-	GM	S
5-simplex	-	-	-	GM

- Global minima are classified comparing energy values.
- Other configurations classified with Hessian of the Lagrangian.
- It is known that problem does not admit local/global maxima.

Conclusions and Future Work

- Formulate critical configurations as polynomial system sols.
- Obtain all critical configurations, and in particular all optimal configurations, for $n \leq 6$, $d \geq 1$.
- Give first exhaustive list of critical configurations and classification for $n = 6$ in S^2.

Conclusions and Future Work

- Formulate critical configurations as polynomial system sols.
- Obtain all critical configurations, and in particular all optimal configurations, for $n \leq 6$, $d \geq 1$.
- Give first exhaustive list of critical configurations and classification for $n = 6$ in S^2.

Pros and Cons of our Approach

✓ Unified framework (previous results are from different articles).

✓ Formulation useful for sphere S^d of any dimension d.

Conclusions and Future Work

- Formulate critical configurations as polynomial system sols.
- Obtain all critical configurations, and in particular all optimal configurations, for $n \leq 6$, $d \geq 1$.
- Give first exhaustive list of critical configurations and classification for $n = 6$ in S^2.

Pros and Cons of our Approach

✓ Unified framework (previous results are from different articles).

✓ Formulation useful for sphere S^d of any dimension d.

× Relies on calculating Gröbner basis in $2\binom{n}{2}$ vars.

Conclusions and Future Work

- Formulate critical configurations as polynomial system sols.
- Obtain all critical configurations, and in particular all optimal configurations, for $n \leq 6$, $d \geq 1$.
- Give first exhaustive list of critical configurations and classification for $n = 6$ in S^2.

Pros and Cons of our Approach

- ✓ Unified framework (previous results are from different articles).
- ✓ Formulation useful for sphere S^d of any dimension d.
- × Relies on calculating Gröbner basis in $2\binom{n}{2}$ vars.

Future Work

- prove optimal configuration for $n = 7$ (numerical candidate $\exists$)
- "simplify" the system of equations, exploiting symmetries.

Gracias por su atención

Code and presentation

`http://www.github.com/matiasvd`

Acknowledgments

- Cluster.UY for the hardware infrastructure.
- ANII and CAP for PhD grants.
- *msolve*, *Macaulay2* and *SymPy* developers for their great work

References

[1] Shub and Smale (1993). Complexity of bezout's theorem: iii. Condition number and packing. Journal of complexity.

[2] Rakhmanov, Saff, Zhou (1995). Electrons on the sphere. In Computational Methods and Function Theory.

[3] Kolushov, Yudin (1997). Extremal dispositions of points on the sphere. Analysis Mathematica.

[4] Smale, Stephen (1998). Mathematical problems for the next century. The mathematical intelligencer.

[5] J.C. Faugère (1999). A new efficient algorithm for computing Gröbner bases (F4). Journal of pure and applied algebra.

[6] Dragnev, Legg, Townsend (2002). Discrete logarithmic energy on the sphere. Pacific journal of mathematics.

[7] Horn, Johnson (2013). Matrix analysis. (2nd ed.). Cambridge university press.

[8] Dragnev (2016). Log-optimal configurations on the sphere. In Modern trends in constructive function theory.

[9] Bétermin, Sandier (2018). Renormalized energy and asymptotic expansion of optimal logarithmic energy on the sphere.

[10] Dragnev, Musin (2023). Log-optimal $(d+2)$-configurations in d-dimensions. Trans. AMS.

[11] Constantineau, et. al (2023). Determination of stable branches of relative equilibria of the N-vortex problem on the sphere.

msolve executions ($n = 6$ points)

AVX512 instructions, 20 threads, Xeon-Gold 6138.

GRevLex monomial order (*msolve* 0.73) - With null CM equations

msolve GB in $\mathbb{Q}$		reduced GB size	
total time	RAM	# pols	# mons
1 hour, 6 minutes	37 GB	2473	xx

Elimination order to project to x_{45} (*msolve* 0.73)

msolve takes 10 hours and 190 GB of RAM.

msolve executions ($n = 6$ points)

AVX512 instructions, 20 threads, Xeon-Gold 6138.

GRevLex monomial order (*msolve* 0.73) - With null CM equations

msolve GB in $\mathbb{Q}$		reduced GB size	
total time	RAM	# pols	# mons
1 hour, 6 minutes	37 GB	2473	xx

Elimination order to project to x_{45} (*msolve* 0.73)

msolve takes 10 hours and 190 GB of RAM.

GRevLex without lifting GB to $\mathbb{Q}$ (*msolve* 0.90)

	msolve GB		reduced GB size	
Null Center of Mass equations	total time	RAM	# pols	# mons
Yes	485 s	3.35 GB	2473	2.133.497
No	433 s	3.36 GB	2473	2.133.497

Configurations for $n = 6$ points in $S^3 \subset \mathbb{R}^4$

Optimal configuration: Real 4

Two equilateral triangles, each inscribed in a copy of S^1 lying in orthogonal spaces.

The other configurations

- Real 1 (SM): analogous to 1:4:1 of S^2: it has the poles, and then the optimal configuration for 4 points in the equatorial sphere.
- Real 3 (S): analogous to 1:5 of S^2: it has a pole, and then the optimal configuration for 5 points in the corresponding sphere.
- Real 2 (S): no analogous on S^2. Has 4 points on the Equator of a sphere and the optimal configuration for 2 points in another sphere. Line through these two points is orthogonal to the plane of the 4 point Equator.

From x_{ij} to w_i

Any solution in the variables w_i has an associated solution $x_{ij} = w_i^T w_j$. Reciprocal is also true in $\mathbb{C}$.

Theorem (Autonne-Takagi factorization [7, Corollary 2.6.6])

If $X \in \mathbb{C}^{n\times n}$ is symmetric, there is a unitary $P \in \mathbb{C}^{n\times n}$, and a non-negative diagonal matrix $D \in \mathbb{R}^{n\times n}$, such that: $X = P^T DP$. Furthermore, the entries of D are the singular values of X.

Corollary

If $X \in \mathbb{C}^{n\times n}$ is symmetric with rank d, and ones on its diagonal, there exists $W \in \mathbb{C}^{d\times n}$, such that: $X = W^T W$, $w_i^T w_i = 1$, $\forall\ i$.

Proof.

As X is symmetric: $X = P^T DP$. Take: $W = \sqrt{\hat{D}}P$; where $\hat{D} \in \mathbb{R}^{d\times n}$ is the submatrix of D with positive singular values. $\square$