Characterization of Logarithmic Fekete Critical Configurations of at Most Six Points in All Dimensions

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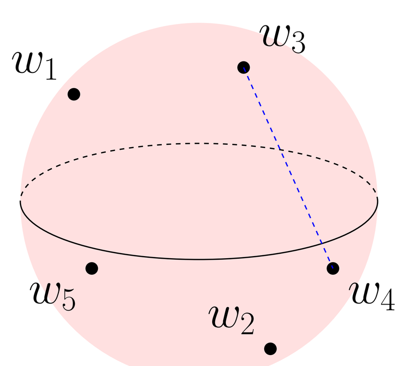
FEKETE PROBLEM

Place n points on the surface of a sphere to maximize the product of euclidean distances between points:

$$\prod_{1\le i<j\le n} \|w_i - w_j\|$$

Equivalently: minimize the logarithmic energy

$$E_{\log}(w) := -\sum_{1\le i<j\le n} \log\left(\|w_i - w_j\|\right)$$



For S^2 the solution is known only for $n \le 6$ and $n = 12$ points.

CONTRIBUTIONS

- Formulate critical configurations as solutions of a polynomial system
- Formulation is useful for any sphere $S^d \subseteq \mathbb{R}^{d+1}$
- For $n \le 6$ points, and all sphere dimensions d:
  - recover previously known results in a unified framework
  - find and classify all critical configurations, including an unknown saddle
- $n = 7$ points in S^2: if global minimum has dipole, it is the configuration 1:5:1

APPROACH

- Determine the number of solutions m of the polynomial system (using Gröbner bases)
- Find as many solutions as possible, until we reach m

POLYNOMIAL SYSTEM

CRITICAL CONFIGURATIONS

Lagrangean of the problem:

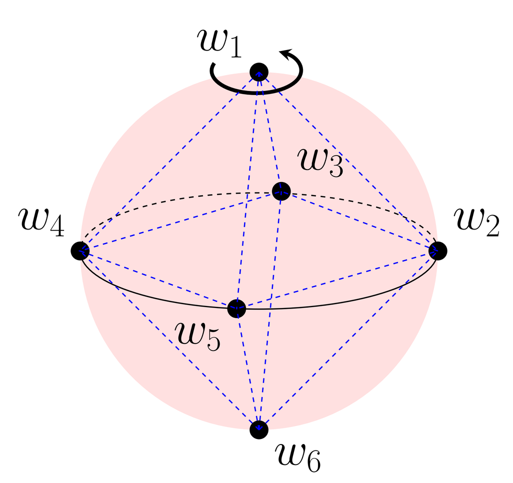
$$L(w,\lambda) := -\sum_{1\le i<j\le n} \log\left(\|w_i - w_j\|^2\right) + \sum_{k=1}^{n} \lambda_k \left(\|w_k\|^2 - 1\right).$$

Optimality conditions:

$$\nabla_{w_i} L = \vec{0}, \quad \|w_i\|^2 = 1 \Leftrightarrow (n-1)\, w_i - \sum_{j=1, j\ne i}^{n} \frac{2(w_i - w_j)}{\|w_i - w_j\|^2} = 0, \ \forall\, i.$$

REMOVING ORTHOGONAL SYMMETRY

- Energy is invariant under rotations $\Rightarrow \infty$ solutions in variables w_i (but we need to count them)
- Use dot product ("angle") as variable:
  $$x_{ij} := w_i^T w_j = \cos(\theta_{ij})$$
- Now rotations give the same solution (in the x_{ij}).



POLYNOMIAL EQUATIONS IN TERMS OF DOT PRODUCTS

$$(n-1)\, x_{ki} - \sum_{j=1, j\ne i}^{n} (x_{ki} - x_{kj}) z_{ij} = 0, \ \forall\, i \ne k, \quad z_{ij}(1 - x_{ij}) = 1.$$

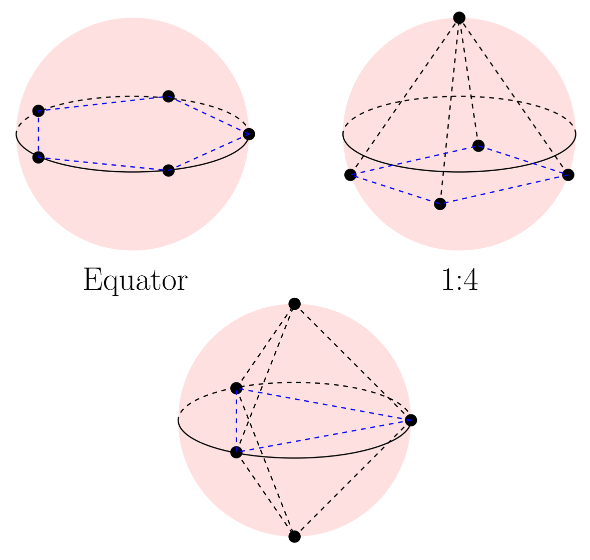
	n	3	4	5	6	7	8
variables (x_{ij}, z_{ij})	$2\binom{n}{2}$	6	12	20	30	42	56
equations	$n(3n-1)/2$	12	22	35	51	70	92

NUMBER OF SOLUTIONS USING GRÖBNER BASIS

Theorem (standard result): Let G be a Gröbner basis of an ideal $I = (f_1, \dots, f_k)$ of the polynomial ring $\mathbb{Q}[x_1, \dots, x_n]$.

1. The system $f_1 = 0, \dots, f_k = 0$ has a **finite number of solutions**, iff: for each variable x_i, G has a polynomial with Leading Monomial $x_i^{m_i}$, for some $m_i \ge 0$.
2. In this case, the **exact number of solutions** (in $\mathbb{C}^n$ and with multiplicities), is the number of the ring monomials not multiple of the Leading Monomials of G.

FIVE POINTS



- Existent solutions (Gröbner): 38.
- Found (proposed): 4 "kind" of solutions, with a total of 38 different permutations.

Configuration	Equator	1:4	1:3:1	4-simplex	Total
$\ne$ permutations	12	15	10	1	38
sphere	S^1	S^2		S^3	

Theorem: the only kind of critical configurations are the ones we proposed

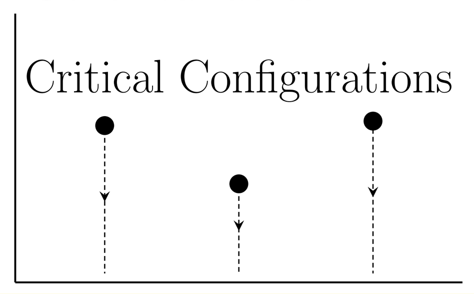
SIX POINTS (Imagination is not enough)

EXISTENT SOLUTIONS

- 938 solutions, including multiplicities (using Leading Monomials of Gröbner basis).
- Now we need to find as many as 938 solutions.
- We split the polynomial system into disjoint cases, one for each value of x_{45}.

POSSIBLE VALUES OF DOT PRODUCT x_{45}

- Project solution set onto variable x_{45}.
- How? Gröbner base of $I \cap \mathbb{Q}[x_{45}]$, using elimination order. This is a PID: $\exists\, h \,/\, I \cap \mathbb{Q}[x_{45}] = (h)$.
- Roots of h are the possible values of x_{45}.



$$h(x_{45}) = x_{45}(x_{45}+1)^2(2x_{45}-1)\dots\left(400x_{45}^4 + 488x_{45}^3 - 111x_{45}^2 - 196x_{45} + 67\right)$$

degree 42, 21 real roots, 16 complex roots

SOLUTIONS FOR EACH VALUE OF x_{45}

For each factor h_i: add $h_i = 0$ to the original system, and find solutions of simplified system

Found solutions: 848 - **Existent**: 938

Difference: 938 - 848 = 90

Complex 1 has multiplicity 2: +90 ✓

We found all possible critical configurations

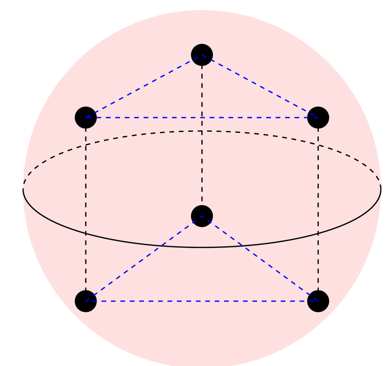
Configuration	**# permutations**	**Sphere**
Equator	60	S^1
1:5	72	S^2
1:4:1	15	
3:3	60+60	
Complex 1	90	
Complex 2	360	
Real 1	15	S^3
Real 2	45	
Real 3	60	
Real 4	10	
5-simplex	1	S^4
Total	**848**	

CLASSIFICATION OF CRITICAL CONFIGURATIONS

Configuration	S^1	S^2	S^3	S^4
Equator	GM	S	S	S
1:5	-	S	S	S
1:4:1	-	GM	S	S
3:3 ($\sqrt{6}$)	-	S	S	S
Real 1	-	-	SM	S
Real 2	-	-	S	S
Real 3	-	-	S	S
Real 4	-	-	GM	S
5-simplex	-	-	-	GM

Saddle (S); Spurious Minima (SM).

- Lowest Energy $\Rightarrow$ Global Minima (GM)
- Other configurations are classified with Hessian of Lagrangian.



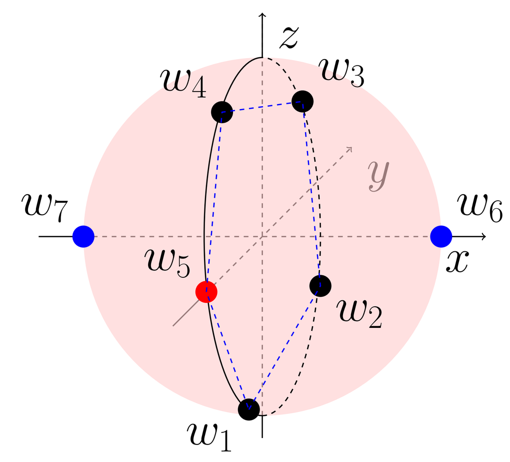
Saddle 3:3

SEVEN POINTS (Resources are not enough)

Configuration 1:5:1 is the conjectured global minima

Theorem: If the global minima in S^2 has a dipole, then it is the configuration 1:5:1.

Proof: Fix two points forming a dipole. We are now able to find a Gröbner basis and count the number of solutions. These are 48, which matches the number of permutations of the critical configuration 1:5:1: $48 = 2 \times 4!$.



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ACKNOWLEDGEMENTS: Cluster.UY for the hardware infrastructure; ANII and CAP for PhD grants; *msolve*, *Macaulay2* and *SymPy* developers for their great work.

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