Elliptic Fekete points by optimizing the Bombieri-Weyl norm

Random and Deterministic Point Configurations
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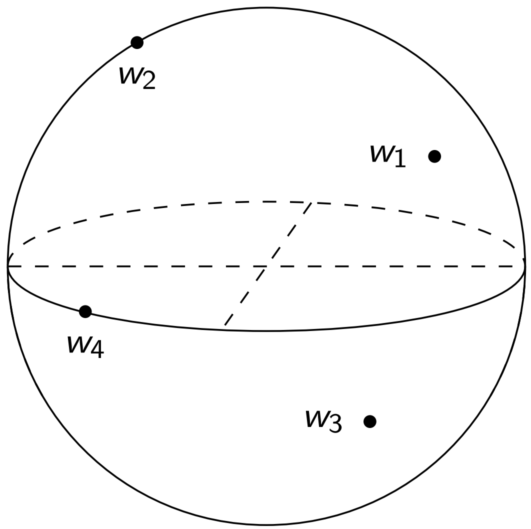
Logarithmic energy

$$E_{\log}(w) := -\log\left(\prod_{i,j=1,i\neq j}^{n} \|w_i - w_j\|_2\right) = -\sum_{i,j=1,i\neq j}^{n} \log\left(\|w_i - w_j\|_2\right)$$

$$w = (w_1, \ldots, w_n)\,,\; w_i \in S^2.$$

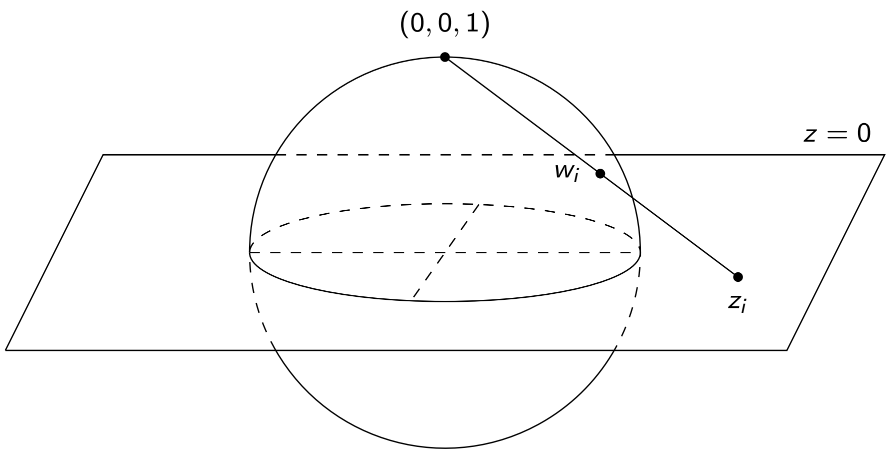


Elliptic Fekete points:

$$\underset{w_i \in S^2}{\operatorname{argmin}}\; E_{\log}(w)$$

Condition number and Bombiery-Weyl norm

$$w_i \in S^2 \to z_i \in \mathbb{C} \to p(z) = (z - z_1)\dots(z - z_n)$$



Condition number: $\mu(p, z_i) = \sqrt{n}\left(\frac{(1 + |z_i|^2)^{\frac{n}{2}-1}}{|p'(z_i)|}\right)\|p\|$

Bombiery-Weyl norm: $p(z) = a_n z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$

$$\|p\| = \sqrt{\sum_{k=0}^{n} \frac{|a_k|^2}{C_k^n}}$$

Log energy, condition number and Bombiery-Weyl norm

Theorem (Armentano, Beltrán, Shub - 2011)

$$E_{\log}(w) = \sum_{i=1}^{n} \log\left(\mu(p, z_i)\right) + n \log\left(\frac{\prod_{i=1}^{n} \sqrt{1+|z_i|^2}}{\|p\|}\right) + C(n)$$

Conjecture (Beltrán - 2020)

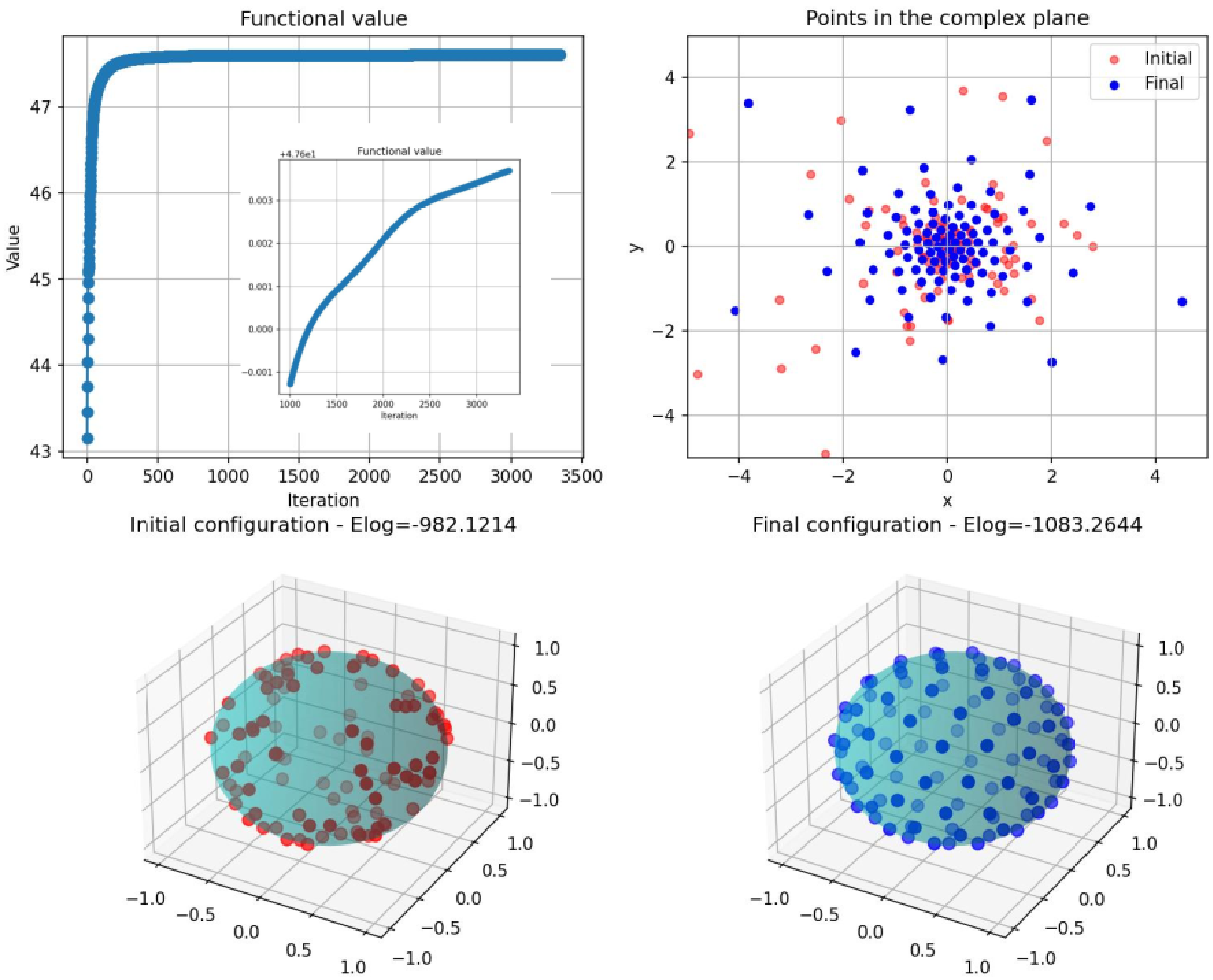
The following problems are equivalent:

$$\operatorname{argmin}_{w_i \in S^2} E_{\log}(w), \quad \operatorname{argmin}_{z_i \in \mathbb{C}} \sum_{i=1}^{n} \log\left(\mu(p, z_i)\right)$$

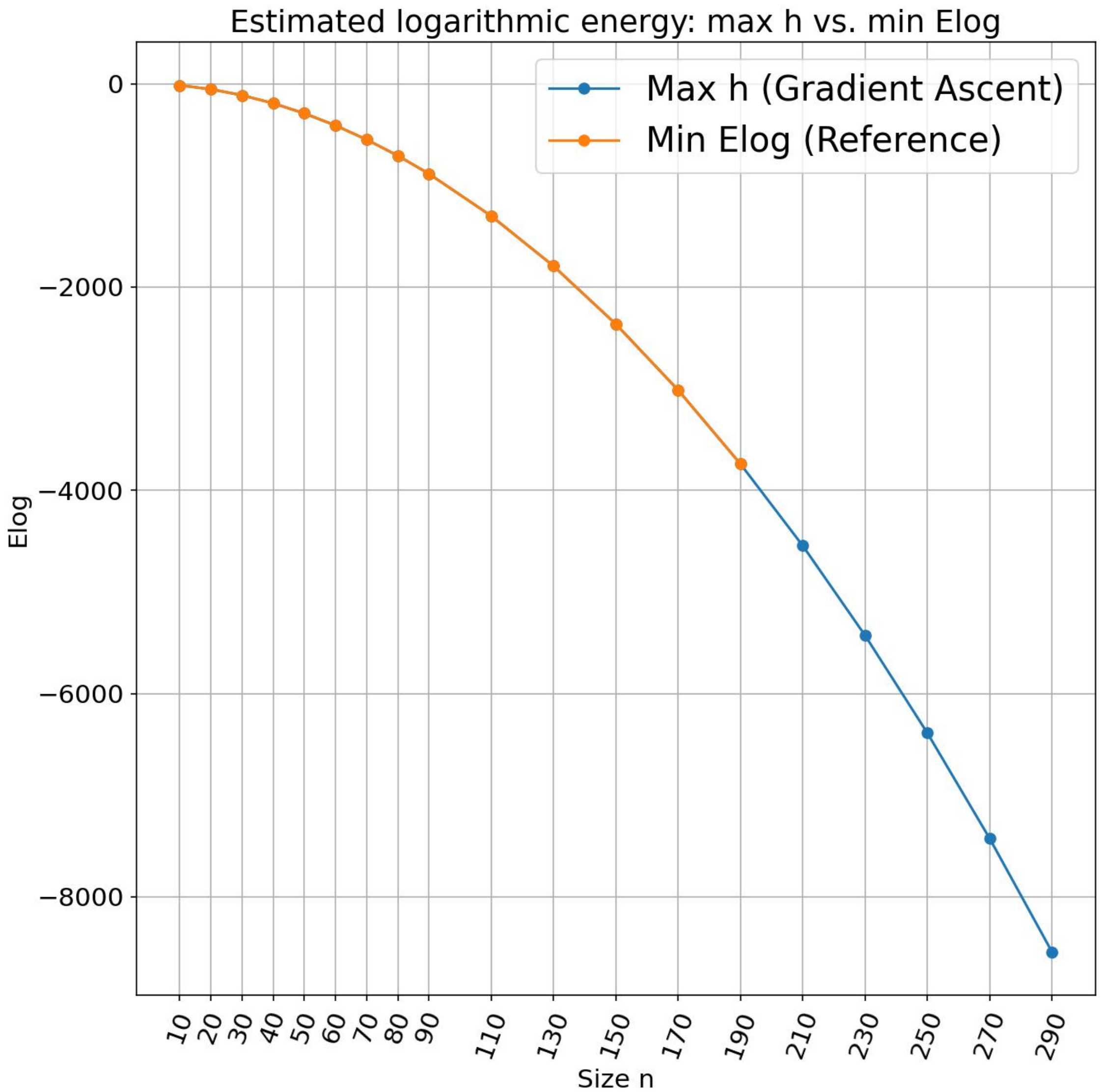
$$\operatorname{argmax}_{z_i \in \mathbb{C}} \log\left(\frac{\prod_{i=1}^{n} \sqrt{1+|z_i|^2}}{\|p\|}\right)$$

Gradient Ascent ($n = 100$)

$$h(z) := \log\left(\frac{\prod_{i=1}^{n}\sqrt{1+|z_i|^2}}{\|p\|}\right), \quad \mathrm{argmax}_{z_i\in\mathbb{C}}\, h(z), \quad z^{(k+1)} = z^{(k)} + \alpha_k \nabla h\left(z^{(k)}\right)$$



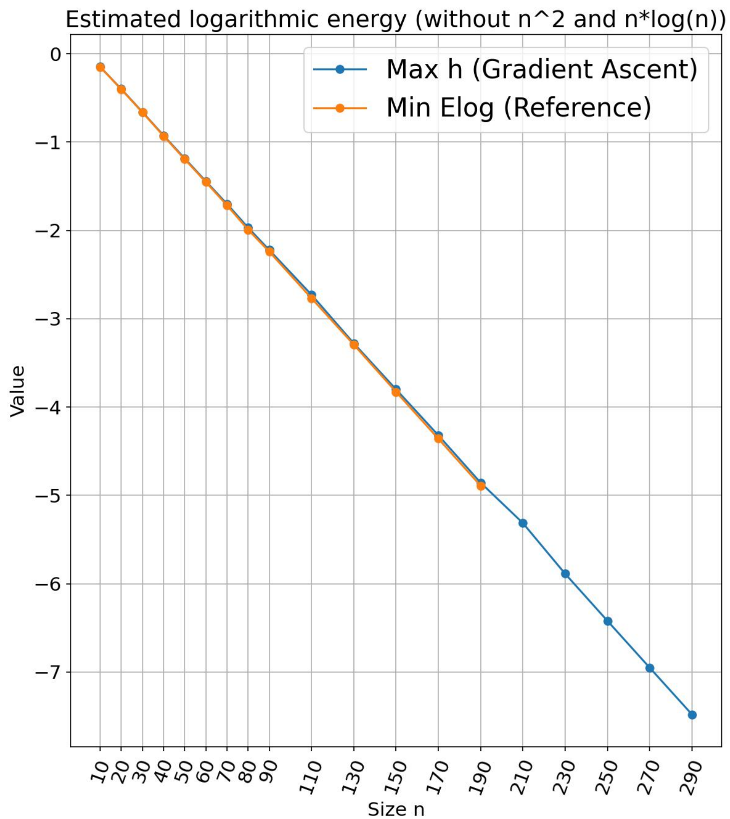
Logarithmic energy of argmax $h(z)$ vs. argmin $E_{\log}(w)$ [1]

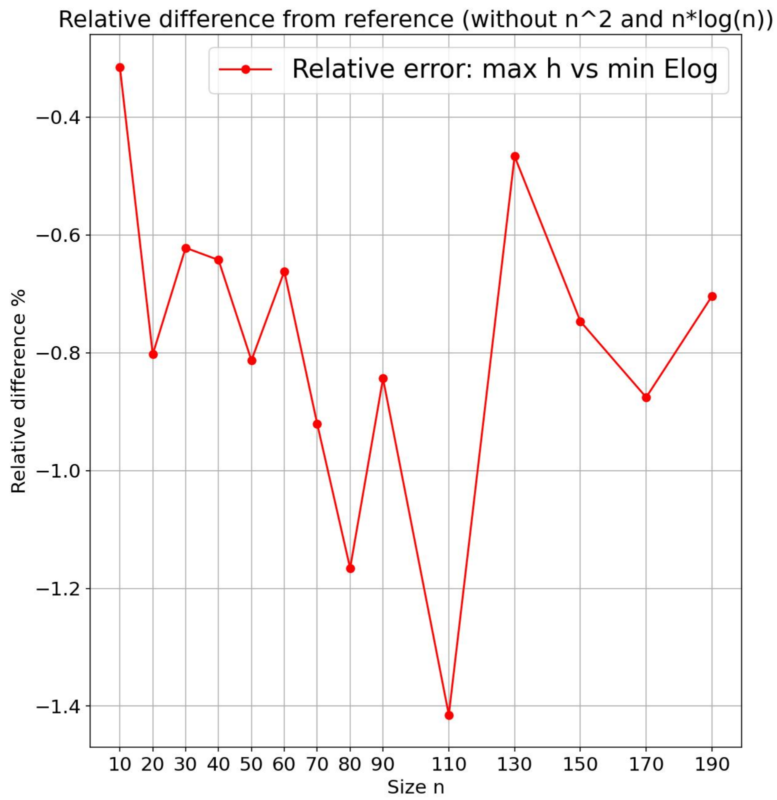


Reference [1]: "Electrons on the sphere" - Zhou, Saff, Rakhmanov (1995)

Logarithmic energy of argmax $h(z)$ vs. argmin $E_{\log}(w)$ [1]

$$E^*_{\log}(n) = Kn^2 - \frac{n\log(n)}{4} + Cn + o(n), \quad K = -\frac{1}{4}\log\left(\frac{4}{e}\right), \; C = ?$$





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Challenges and future work

$$E_{\log}(w) = \sum_{i=1}^{n} \log\left(\mu(p, z_i)\right) + n \log\left(\frac{\prod_{i=1}^{n}\sqrt{1+|z_i|^2}}{\|p\|}\right) + C(n)$$

1. We analyzed argmax of $h(z) = \log\left(\frac{\prod_{i=1}^{n}\sqrt{1+|z_i|^2}}{\|p\|}\right)$; $n \leq 300$
2. Analyze argmin of

$$g(z) := \sum_{i=1}^{n} \log\left(\mu(p, z_i)\right) = \sum_{i=1}^{n} \log\left(\sqrt{n}\left(\frac{\left(1+|z_i|^2\right)^{\frac{n}{2}-1}}{|p'(z_i)|}\right)\|p\|\right)$$

3. g and h depend on $\|p\| = \sqrt{\sum_{k=0}^{n} \frac{|a_k|^2}{C_k^n}}$. But gradient is w.r.t. z
4. Lower than $O(n^3)$ gradient $\Rightarrow$ bigger n and smaller tolerance