PARTIAL HYPERBOLICITY AND ERGODICITY IN DIMENSION THREE.

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Abstract. In [15] the authors proved the Pugh-Shub conjecture for partially hyperbolic diffeomorphisms with 1-dimensional center, i.e. stable ergodic diffeomorphism are dense among the partially hyperbolic ones. In this work we address the issue of giving a more accurate description of this abundance of ergodicity. In particular, we give the first examples of manifolds in which *all* conservative partially hyperbolic diffeomorphisms are ergodic.

1. Introduction

A diffeomorphism $f : M \to M$ of a closed smooth manifold M is partially hyperbolic if TM splits into three invariant bundles such that one of them is contracting, the other is expanding, and the third, called the center bundle, has an intermediate behavior, that is, not as contracting as the first, nor as expanding as the second (see the Section 3 for a precise definition). The first and second bundles are called strong bundles.

A central point in dynamics is to find conditions that guarantee ergodicity. In 1994, the pioneer work of Grayson, Pugh and Shub [10] suggested that partial hyperbolicity could be "essentially" a sufficient condition for ergodicity. Indeed, soon afterwards, Pugh and Shub conjectured that stable ergodicity (open sets of ergodic diffeomorphisms) was dense among partially hyperbolic systems. They proposed as an important tool the accessibility property (see also the previous work by Brin and Pesin [2]): f is accessible if any two points of M can be joined by a curve that is a finite union of arcs tangent to the strong bundles. Essential accessibility is the weaker property that any two measurable sets of positive measure can be joined by such a curve. In fact, accessibility will play a key role in this work.

Pugh and Shub split their Conjecture into two sub-conjectures: (1) essential accessibility implies ergodicity, (2) accessibility contains an open and dense set of partially hyperbolic diffeomorphisms.

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Many advances have been made since in the ergodic theory of partially hyperbolic diffeomorphisms. In particular, there is a result by Burns and Wilkinson [4] proving that essential accessibility plus a bunching condition (trivially satisfied if center bundle is one dimensional) implies ergodicity. There is also a result by the authors [15] obtaining the complete Pugh-Shub conjecture for one-dimensional center bundle. See [17] for a recent survey on the subject.

We have therefore that almost all one-dimensional center bundled partially hyperbolic diffeomorphisms are ergodic. This means that the *non-ergodic* partially hyperbolic systems are very few. Can we describe them? Concretely,

QUESTION 1.1. *Which manifolds support a non-ergodic partially hyperbolic diffeomorphism?*

In this work we address this question on three dimensional manifolds. We study the sets of points that can be joined by paths everywhere tangent to the strong bundles (accessibility classes), and arrive, using tools of geometry of laminations, to the somewhat surprising conclusion that there are strong obstructions to the non-ergodicity of a partially hyperbolic diffeomorphism. See Theorem 1.3.

This gave us enough evidence to conjecture the following:

CONJECTURE 1.2. *The only orientable manifolds supporting non-ergodic partially hyperbolic diffeomorphisms in dimension 3 are the mapping tori of diffeomorphisms of surfaces which commute with Anosov diffeomorphisms.*

Specifically, they are (1) the mapping tori of Anosov diffeomorphisms of $\mathbb{T}^2$*, (2)* $\mathbb{T}^3$*, and (3) the mapping torus of* $-id$ *where* $id : \mathbb{T}^2 \to \mathbb{T}^2$ *is the identity map on the 2-torus.*

Indeed, we believe that for 3-manifolds, all partially hyperbolic diffeomorphisms are ergodic, unless the manifold is one of the listed above.

We are able to prove this conjecture when the fundamental group of the manifold is nilpotent:

THEOREM 1.3. *All the conservative* C^2 *partially hyperbolic diffeomorphisms of a compact orientable 3-manifolds with nilpotent fundamental group are ergodic, unless the manifold is* $\mathbb{T}^3$.

A paradigmatic example is the following. Let M be the mapping torus of $A_k : \mathbb{T}^2 \to \mathbb{T}^2$, where A_k is the automorphism given by the matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, k a non-zero integer. That is, M is the quotient of $\mathbb{T}^2 \times [0,1]$ by the relation $\sim$, where $(x,1) \sim (A_k x, 0)$. The manifold M has nilpotent fundamental group; in fact, it is a nilmanifold (see Section 7). Theorem 1.3 then implies that all conservative partially hyperbolic diffeomorphisms of M are ergodic.

Let us see that the above case, namely the case of nilmanifolds, is the only one where Theorem 1.3 is substantial. The Geometrization Conjecture, gives, after Perelman's work:

THEOREM 1.4. *If M is a compact orientable manifold with nilpotent fundamental group, then either M is a nilmanifold or else it is finitely covered by $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{S}^1$.*

The second case mentioned in Theorem 1.4 is ruled out by a remarkable result by Burago and Ivanov:

THEOREM 1.5 ([3]). *There are no partially hyperbolic diffeomorphisms in $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{S}^1$.*

On the other hand, let us note that Theorem 1.3 is not about the empty set, see for instance [19], where Sacksteder presents examples of ergodic affine diffeomorphisms of nilmanifolds that are partially hyperbolic. In [17] there is a detailed treatment of these examples.

The proofs of the theorems above involve deep results of the geometry of codimension one foliations of 3-manifolds. In Section 2 we shall include, for completeness, the basic facts and definitions that we shall be using in this paper. However, the interested reader is strongly encouraged to consult [5], [6] and [13] for a well organized and complete introduction to the subject.

We shall split the preliminaries of this paper into two sections. Section 2 is devoted to the basic facts and definitions about geometry of foliations that will be used. Section 3 will cover the basic facts and definitions about the dynamics part of this paper.

Now let us describe the structure of our proof. In the first place, it follows from the results in [4, 15] that accessibility implies ergodicity. So, our strategy will be to prove that all partially hyperbolic diffeomorphisms of compact 3-manifolds with nilpotent fundamental group different from $\mathbb{T}^3$ satisfy the accessibility property.

In dimension 3, and in fact, whenever the center bundle is 1-dimensional, the non-open accessibility classes are codimension one immersed manifolds [15]; the set of non-open accessibility classes is a compact set *laminated* by the accessibility classes (see Section 2 for definitions). So, either f has the accessibility property or else there is a non-trivial lamination formed by non-open accessibility classes.

Let us first assume that the lamination is not a foliation (i.e. does not cover the whole manifold). Then we will show that it either extends to a true foliation without compact leaves, or else it contains a leaf that is a periodic 2-torus with Anosov dynamics. In the first case, we have that the boundary leaves of the lamination contain a dense set of periodic points [15], and that their fundamental group injects in the fundamental group of the manifold, which is nilpotent. Therefore, it will follow that the boundary leaves of the lamination are, in fact, tori, which is a contradiction. Henceforth, in case the non-open accessibility classes are a strict lamination, that is, not a foliation, there is a periodic 2-torus with Anosov dynamics. This is done in Section 4.

We shall call any embedded 2-torus admitting an Anosov dynamics extendable to the whole manifold, an *Anosov torus*. That is, $T \subset M$ is an Anosov torus if there exists $h : M \to M$ such that $h|_T$ is Anosov. In Section 5 we prove that Anosov tori are incompressible. The fundamental group of an Anosov torus hence injects in the (nilpotent) fundamental group of our manifold. In Section 7, Corollary 7.3, we prove that the only manifold with nilpotent fundamental group admitting Anosov tori is, in fact, $\mathbb{T}^3$.

So, we have arrived to the conclusion that if there are non-open accessibility classes, they must foliate the whole manifold. Let us see that this foliation can not have compact leaves. Observe that any such compact leaf must be a 2-torus. So, we have three possibilities: (1) there is an Anosov torus, (2) the set of compact leaves forms a strict non-trivial lamination, (3) the manifold is foliated by 2-tori. The first case has just been ruled out. In the second case, we would have that the boundary leaves contain a dense set of periodic points, as stated above, and hence they would be Anosov tori again, which is impossible. Finally, in the third case, we conclude that the manifold is a fibration of tori over $\mathbb{S}^1$ (see Section 8, page 20). This can only occur, in our setting, if the manifold is $\mathbb{T}^3$.

In this way we have that a conservative partially hyperbolic diffeomorphism of a compact 3-manifold with nilpotent fundamental group different from $\mathbb{T}^3$ has the accessibility property or else the accessibility classes form a codimension one foliation of the manifold without compact leaves. The rest of the paper is devoted to excluding this last possibility.

Indeed, in case there is a codimension one invariant foliation by accessibility classes it is shown, using subtle results of Plante and Roussarie, and the dynamics of f that this foliation consists of "parallel" cylinders (Section 8).

On the other hand, we show in Section 7 that f should be semi-conjugate to an Anosov diffeomorphism of the 2-torus. The fact that there is a foliation tangent to $E^s \oplus E^u$ is used in the construction of this semi-conjugacy. In Section 8 we show that this leads to a contradiction.

The following theorem summarizes many of the results in this paper. See definitions in Section 2:

THEOREM 1.6. *Let $f : M \to M$ be a conservative partially hyperbolic diffeomorphism of an orientable 3-manifold M. Suppose that the bundles E^σ are also orientable, $\sigma = s, c, u$, and that f is not accessible. Then one of the following possibilities holds:*

1. *there is an f-periodic incompressible torus tangent to $E^s \oplus E^u$.*
2. *there is an f-invariant lamination $\emptyset \neq \Gamma(f) \neq M$ tangent to $E^s \oplus E^u$ that trivially extends to a (not necessarily invariant) foliation without compact leaves of M. Moreover, the boundary leaves of $\Gamma(f)$ are periodic, have Anosov dynamics and dense periodic points.*
3. *there is a Reebless invariant foliation tangent to $E^s \oplus E^u$.*

The assumption on the orientability of the bundles and M is not essential, in fact, it can be achieved by a finite covering. We do not know of any example satisfying (2) in the theorem above (see Question 1.7). The proof of Theorem 1.6 appears at the end of Section 5.

In [17] (Theorem 4.11) we had announced that $\Gamma(f)$ has always an invariant torus leaf in case it is a strict lamination. Unfortunately our proof has a gap and the following question (Problem 22 and commentary below of [17]), up to our knowledge, remains open even in the codimension one case.

Question 1.7. *Let $f : N \to N$ be an Anosov diffeomorphism on a complete Riemannian manifold N. Is it true that if $\Omega(f) = N$ then N is compact?*

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2. Geometric preliminaries

In this section we state several definitions and concepts that will be useful in the rest of this paper. From now on, M will be a compact connected Riemannian 3-manifold.

A *lamination* is a compact set $\Lambda \subset M$ that can be covered by open charts $U \subset \Lambda$ with a local product structure $\phi : U \to \mathbb{R}^n \times T$, where T is a locally compact subset of $\mathbb{R}^k$. On the overlaps $U_\alpha \cap U_\beta$, the transition functions $\phi_\beta \circ \phi_\alpha^{-1} : \mathbb{R}^n \times T \to \mathbb{R}^n \times T$ are homeomorphisms and take the form:

$$\phi_\beta \circ \phi_\alpha^{-1}(u, v) = (l_{\alpha\beta}(u, v), t_{\alpha\beta}(v)),$$

where $l_{\alpha\beta}$ are C^1 with respect to the u variable. No differentiability is required in the transverse direction T. The sets $\phi^{-1}(\mathbb{R}^n \times \{t\})$ are called *plaques*. Each point x of a lamination belongs to a maximal connected injectively immersed n-submanifold, called the *leaf* of x in L. The leaves are union of plaques. Observe that the leaves are C^1, but vary only continuously. The number n is the *dimension* of the lamination. If $n = \dim M - 1$, we say Λ is a *codimension-one* lamination. The set L is an *f-invariant lamination* if it is a lamination such that f takes leaves into leaves.

We call a lamination a *foliation* if $\Lambda = M$. In this case, we shall denote by $\mathcal{F}$ the set of leaves. In principle, we shall not assume any transverse differentiability. However, in case $l_{\alpha\beta}$ is C^r with respect to the v variable, we shall say that the foliation is C^r. Note that even purely C^0 codimension-one foliations admit a transverse 1-dimensional foliation (see Siebenmann [20], Solodov [21]). This allows to translate many local deformation arguments, usually given in the C^2 category, into the C^0 category. In particular, Theorems 2.1 and 2.3, which were originally formulated for C^2 foliations hold in the C^0 case. We shall say that a codimension-one foliation $\mathcal{F}$, is *transversely orientable* if the transverse 1-dimensional foliation mentioned above is orientable. An *invariant foliation* is a foliation that is an invariant lamination.

Let Λ be a codimension-one lamination that is not a foliation. A *complementary region* V is a component of $M \setminus \Lambda$. A *closed complementary region* $\hat{V}$ is the metric completion of a complementary region V with the path metric induced by the Riemannian metric, the distance between two points being the infimum of lengths of paths in V connecting them. A closed complementary region is independent of the metric. Note that they are not necessarily compact. If Λ does not have compact leaves, then every closed complementary region decomposes into a compact *gut* piece and non-compact *interstitial regions* which are I-bundles over non-compact surfaces, and get thinner and thinner as they go away from the guts (see [13] or [9]). The interstitial regions meet the guts along annuli. The decomposition into interstitial regions and guts is unique up to isotopy. Moreover, one can take the interstitial regions as thin as one wishes.

A *boundary leaf* is a leaf corresponding to a component of ∂V, for V a closed complementary region. That is, a leaf is a non-boundary leaf if it is not contained in a closed complementary region.

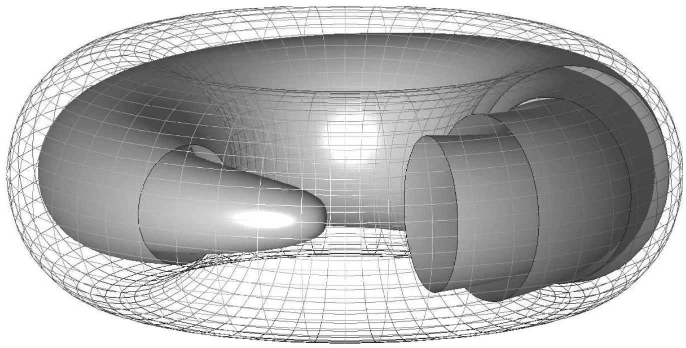

FIGURE 1. A Reeb component

The geometry of codimension-one foliations is deeply related to the topology of the manifold that supports them. The following subset of a foliation is important in their description. A *Reeb component* is a solid torus whose interior is foliated by planes transverse to the core the solid torus, such that each leaf limits on the boundary torus, which is also a leaf (see Figure 1). A foliation that has no Reeb components is called *Reebless*.

The following theorems show better the above mentioned relation:

THEOREM 2.1 (Novikov). *Let M be a compact orientable 3-manifold and $\mathcal{F}$ a transversely orientable codimension-one foliation. Then each of the following implies that $\mathcal{F}$ has a Reeb component:*

(1) *There is a closed, nullhomotopic transversal to* $\mathcal{F}$
(2) *There is a leaf* L *in* $\mathcal{F}$ *such that* $\pi_1(L)$ *does not inject in* $\pi_1(M)$

The statement of this theorem can be found, for instance, in [6] Theorems 9.1.3. 9.1.4., p.288. We shall also use the following theorem

THEOREM 2.2 (Haefliger). *Let* Λ *be a lamination in* M. *Then the set of points belonging to compact leaves is compact.*

This theorem was originally formulated for foliations [11]. However, it also holds for laminations, see for instance [13].

We have the following consequence of Novikov's Theorem about Reebless foliations. This theorem is stated in [18] as Corollary 2 on page 44.

THEOREM 2.3. *If* M *is a compact 3-manifold and* $\mathcal{F}$ *is a transversely orientable codimension-one Reebless foliation, then either* $\mathcal{F}$ *is the product foliation of* $\mathbb{S}^2 \times \mathbb{S}^1$, *or* $\tilde{\mathcal{F}}$, *the foliation induced by* $\mathcal{F}$ *on the universal cover* $\tilde{M}$ *of* M, *is a foliation by planes* $\mathbb{R}^2$. *In particular, if* $M \neq \mathbb{S}^2 \times \mathbb{S}^1$ *then* M *is irreducible.*

This theorem was originally stated for C^2 foliations, but it also holds for C^0 foliations, due to Siebenmann's theorem mentioned above.

3. DYNAMIC PRELIMINARIES

Throughout this paper we shall work with a *partially hyperbolic diffeomorphism* f, that is, a diffeomorphism admitting a non trivial Tf-invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$, such that all unit vectors $v^\sigma \in E^\sigma_x$ $(\sigma = s, c, u)$ with $x \in M$ verify:

$$\|T_x f v^s\| < \|T_x f v^c\| < \|T_x f v^u\|$$

for some suitable Riemannian metric. f also must satisfy that $\|Tf|_{E^s}\| < 1$ and $\|Tf^{-1}|_{E^u}\| < 1$.

We shall also assume that f is *conservative*, i.e. it preserves Lebesgue measure associated to a smooth volume form.

It is a known fact that there are foliations $\mathcal{W}^\sigma$ tangent to the distributions E^σ for $\sigma = s, u$ (see for instance [2]). The leaf of $\mathcal{W}^\sigma$ containing x will be called $W^\sigma(x)$, for $\sigma = s, u$. The connected component of x in the intersection of $W^s(x)$ with a small ε-ball centered at x is the *ε-local stable manifold of* x, and is denoted by $W^s_\varepsilon(x)$.

In general it is not true that there is a foliation tangent to E^c. It is still unknown if there is such a foliation in case $\dim E^c = 1$. However, in Proposition 3.4 of [1] it is shown that if $\dim E^c = 1$, then f is *weakly dynamically coherent*. This means, for each $x \in M$ there are complete immersed C^1 manifolds which contain x and are everywhere tangent to E^c, E^{cs} and E^{cu}, respectively. We will call *center curve* any curve which is everywhere tangent to E^c. Moreover, we will use the following fact:

PROPOSITION 3.1 ([1]). *If γ is a center curve through x, then*

$$W^s_\varepsilon(\gamma) = \bigcup_{y\in\gamma} W^s_\varepsilon(y) \qquad \textit{and} \qquad W^u_\varepsilon(\gamma) = \bigcup_{y\in\gamma} W^u_\varepsilon(y)$$

are C^1 immersed manifolds everywhere tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$ respectively.

We shall say that a set X is *s-saturated* or *u-saturated* if it is a union of leaves of the strong foliations $\mathcal{W}^s$ or $\mathcal{W}^u$ respectively. We also say that X is su-saturated if it is both s- and u-saturated. The *accessibility class* $AC(x)$ of the point $x \in M$ is the minimal su-saturated set containing x. Note that the accessibility classes form a partition of M. If there is some $x \in M$ whose accessibility class is M, then the diffeomorphism f is said to have the *accessibility property*. This is equivalent to say that any two points of M can be joined by a path which is piecewise tangent to E^s or to E^u.

The theorem below relates accessibility with ergodicity. In fact it is proven in a more general setting, but we shall use the following formulation:

THEOREM 3.2 ([4],[15]). *If f is a C^2 conservative partially hyperbolic diffeomorphism with the accessibility property and* $\dim E^c = 1$, *then f is ergodic.*

We will prove that there are manifolds whose topology implies the accessibility property for all partially hyperbolic diffeomorphisms. In these manifolds, all partially hyperbolic diffeomorphisms are ergodic.

We will focus on the openness of the accessibility classes. Note that the accessibility classes form a partition of M. If all of them are open then, in fact, f has the accessibility property. We will call $U(f) = \{x \in M; AC(x) \text{ is open}\}$ and $\Gamma(f) = M \setminus U(f)$. Note that f has the accessibility property if and only if $\Gamma(f) = \emptyset$. We have the following property of non-open accessibility classes:

PROPOSITION 3.3 ([15]). *The set $\Gamma(f)$ is a codimension-one lamination, having the accessibility classes as leaves.*

In fact, any compact su-saturated subset of $\Gamma(f)$ is a lamination.

The above proposition is Proposition A.3. of [15]. The fact that the leaves of $\Gamma(f)$ are C^1 may be found in [7]. The following proposition is Proposition A.5 of [15]:

PROPOSITION 3.4 ([15]). *If Λ is an invariant sub-lamination of $\Gamma(f)$, then each boundary leaf of Λ is periodic and the periodic points are dense in it (with the induced topology).*

Moreover, the stable and unstable manifolds of each periodic point are dense in each plaque of a boundary leaf of Λ

Observe that the proof of Proposition A.5 of [15] shows in fact that periodic points are dense in the accessibility classes of the boundary leaves of V endowed with its intrinsic topology. In other words, periodic points are dense in each plaque of the boundary leaves of V.

We shall also use the following theorem by Brin, Burago and Ivanov, whose proof is in [1], after Proposition 2.1.

THEOREM 3.5 ([1]). *If $f : M^3 \to M^3$ is a partially hyperbolic diffeomorphism, and there is an open set V foliated by center-unstable leaves, then there cannot be a closed center-unstable leaf bounding a solid torus in V.*

4. THE su-LAMINATION $\Gamma(f)$

Let f be a partially hyperbolic diffeomorphism of a compact 3-manifold M. From Section 3 it follows that we have three possibilities: (1) f has the accessibility property, (2) the set of non-open accessibility classes is a strict lamination, $\emptyset \subsetneq \Gamma(f) \subsetneq M$ or (3) the set of non-open accessibility classes foliates M: $\Gamma(f) = M$. Our goal is to discard possibilities (2) and (3).

Now, we shall distinguish two possible cases in situations (2) and (3):

(a) the lamination $\Gamma(f)$ does not contain compact leaves
(b) the lamination $\Gamma(f)$ contains compact leaves

In this section we deal with the case (2a). In fact, for our purposes it will be sufficient to assume that there exists an f-invariant sub-lamination Λ of $\Gamma(f)$ without compact leaves. Section 5 treats the cases (2b) and (3b). Section 8 treats the case (3a).

In this section, we will prove that the complement of Λ consists of I-bundles. To this end, we shall assume that the bundles E^σ ($\sigma = s, c, u$) and the manifold M are orientable (we can achieve this by considering a finite covering).

THEOREM 4.1. *If $\emptyset \subsetneq \Lambda \subset \Gamma(f)$ is an orientable and transversely orientable f-invariant sub-lamination without compact leaves such that $\Lambda \neq M$, then all closed complementary regions of Λ are I-bundles.*

We will prove Theorem 4.1 by showing:

PROPOSITION 4.2. *Let $\Lambda \subset \Gamma(f)$ be a nonempty f-invariant sub-lamination without compact leaves. Then E^c is uniquely integrable in the closed complementary regions of Λ.*

Let us consider $\hat{V}$ a closed complementary region of Λ, and call $\mathcal{I}(V)$ the union of all interstitial regions of V and $\mathcal{G}(V)$ the gut of $\hat{V}$ (see Section 2), so that

$$\hat{V} = \mathcal{I}(V) \cup \mathcal{G}(V).$$

In the Remark 3.7 of [16] it is proved:

LEMMA 4.3 ([16]). *There exists $\delta > 0$ such that if E^c is not uniquely integrable at x and γ is a central curve through x, then there exists $N > 0$ for which the length of $f^n(\gamma)$ or of $f^{-n}(\gamma)$ is greater than δ for all $n > N$.*

Proof of Proposition 4.2. Let $\mathcal{I}(V)$ be so thin that any center curve in $\mathcal{I}(V)$ meets two boundary leaves of $\mathcal{I}(V)$ and is bounded from above by $\delta/2$. This is possible for a small $\delta > 0$, since E^c is transverse to the boundary leaves, and $\mathcal{I}(V)$ can be chosen arbitrarily thin (see Section 2).

Let y be a recurrent point in the interior of $\mathcal{I}(V)$. Take γ a center curve through y. If E^c were not uniquely integrable at y, then we would find a sufficiently large iterate $k > 0$ of f such that $f^k(y)$ is in the interior of $\mathcal{I}(V)$ and the length of $f^k(\gamma)$ is greater than δ (Lemma 4.3), which contradicts our choice of $\mathcal{I}(V)$.

Note that this implies that E^c is uniquely integrable at each point z in $W^c(y)$, where y is a recurrent point in the interior of $\mathcal{I}(V)$. If E^c were not uniquely integrable at $z \in W^c(y)$, then we would have on one hand that for k as above $f^k(z) \in W^c(f^k(y)) \subset \mathcal{I}(V)$, and from Lemma 4.3 we would have that length of $f^k(W^c(y))$ is greater than δ, a contradiction.

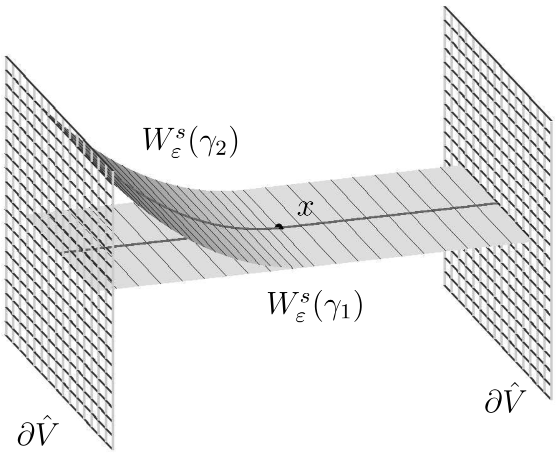

FIGURE 2. A point at which E^c is non uniquely integrable

Now, let us suppose that at $x \in \mathcal{I}(V)$ there are two center curves γ_1 and γ_2. We may assume that they are not contained in the same center-stable leaf, otherwise, we consider center-unstable leaves instead. Consider two different center-unstable leaves CU_1 and CU_2 very close to x. Then $\partial\hat{V}$, $W^s_\varepsilon(\gamma_1)$, $W^s_\varepsilon(\gamma_2)$, CU_1 and CU_2 enclose an open subset of $\mathcal{I}(V)$. Take a recurrent point y in this open subset. Its center curve $W^c(y)$ meets two boundary leaves. This forces $W^c(y)$ to exit through $W^s_\varepsilon(\gamma_1)$, $W^s_\varepsilon(\gamma_2)$, CU_1 or CU_2. The point of intersection of $W^c(y)$ with any of these four sets is a point of non-unique integrability of E^c, due to Proposition 3.1.

But this contradicts what was proven above. Hence E^c is uniquely integrable in $\mathcal{I}(V)$. See Figure 2.

Let $\mathcal{C}$ be an open set in $\mathcal{I}(V)$, consisting of central leaves which join two boundary leaves of $\hat{V}$. And let us prove that

$$\mathcal{D} = \bigcup_{n\in\mathbb{Z}} f^n(\mathcal{C}) \tag{4.1}$$

is dense in $\hat{V}$. We begin by observing that $\overline{\mathcal{D}}$, where the closure is taken in the completion $\hat{V}$ of V, is an su-saturated set. Indeed, let y be a recurrent point in the interior of $\mathcal{D}$. There exists a sequence $f^{k_n}(y) \to y$, and a fixed $\varepsilon > 0$ such that $W^s_\varepsilon(f^{k_n}(y))$ is in $\mathcal{D}$. Then $W^s(y) = \bigcup_{n>0} f^{-k_n}(W^s_\varepsilon(f^{k_n}(y)))$ is in $\mathcal{D}$. Now, any point $z \in \overline{\mathcal{D}}$ can be approximated by a sequence y_n of recurrent points. For each fixed $K > 0$, the stable manifold $W^s_K(z)$ is the limit of the stable manifolds $W^s_K(y_n)$, and hence it belongs to $\overline{\mathcal{D}}$. Taking $K \to \infty$ we get the s-saturation of $\overline{\mathcal{D}}$. u-saturation of $\overline{\mathcal{D}}$ is obtained analogously, so we obtain that $\overline{\mathcal{D}}$ is su-saturated.

The boundary of $\overline{\mathcal{D}}$ in $\hat{V}$ consists of leaves of $\Gamma(f)$. Now, if $\overline{\mathcal{D}}$ were not all $\hat{V}$, then there would be one of such leaves, L that is contained in V. L is approximated by points in $\mathcal{D}$. The center leaf of any such point must meet L, due to transversality. But on the other hand, this center leaf joins two boundary leaves of $\hat{V}$, and is fully contained in $\mathcal{D}$, which is open. Hence the interior of $\mathcal{D}$ would meet L, a contradiction. This proves that any open set defined as in Equation (4.1) is dense.

So, there is an open and dense set of points in $\hat{V}$ for which there is a unique center-stable plaque, and moreover, such that $E^s \oplus E^c$ is uniquely integrable at each point of its plaque. Take a small ball B such that each center-stable plaque of a point in this ball cuts all unstable arcs of the ball in two connected components. We can establish an order in the unstable arcs that is coherent in the ball. Observe that two different plaques cut all unstable arcs in the ball preserving this order, unless they meet at a point. If two different center-stable plaques P_1, P_2 meet at a point B, then there is a component in $B \setminus (P_1 \cup P_2)$ where all unstable arcs are bounded from below by, say P_1, and from above by P_2. All plaques P in this component verify $P_1 \leq P \leq P_2$. However $P_1 = P_2$ at x, this means that there is an open set of points whose center-stable plaque passes through x. This is a contradiction.

Therefore, we get that the bundle $E^s \oplus E^c$ is uniquely integrable in $\hat{V}$. Analogously, we get the unique integrability of $E^c \oplus E^u$, whence the unique integrability of E^c follows. □

The following statement follows, with minor modifications, from the proof above:

LEMMA 4.4. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism. If U is an open invariant set such that $U \subset \Omega(f)$, then the closure of U is su-saturated.*

Let us observe that if $\hat{V}$ is connected then there are only two boundary leaves of $\hat{V}$. This follows from density of any set defined as in Equation (4.1). Also, since periodic points are dense in the boundary leaves due to Proposition 3.4, there is an iterate of f that fixes all connected components of $\hat{V}$, so we will assume when proving Theorem 4.1 that $\hat{V}$ is connected and has two boundary leaves L_0 and L_1.

Proof of Theorem 4.1. The strategy will be to show that all center leaves in $\hat{V}$ meet both L_0 and L_1. Let p be a periodic point in $L_0 \cap \mathcal{I}(V)$. Then its center leaf meets L_1, and the same happens for all points in its stable and unstable manifolds. Now stable and unstable manifolds of a periodic point are dense in each plaque of L_0 (Proposition 3.4). So the set of points in L_0 whose center leaf does not reach L_1 is contained in a totally disconnected set.

Let us suppose that x_0 is a point in L_0 whose center leaf does not reach L_1. Consider a small neighborhood SU_0 of x_0 in a plaque of L_0, such that its boundary is contained in the union of the stable manifold and the unstable manifold of a periodic point of L_0. The union of center leaves of SU_0 gives us a (possibly non-compact) manifold in $\hat{V}$, whose boundary is a prism, consisting of two stable-unstable discs SU_0 and SU_1 in L_0, L_1; two center-stable discs CS_0 and CS_1, and two center-unstable discs CU_0 and CU_1. This manifold with boundary is contained in $\mathcal{G}(V)$, the compact gut of $\hat{V}$. Indeed, the length of the center leaf of x_0 is greater than 2δ, and so are the lengths of the center leaves of all points in SU_0. But, as we have stated at the beginning of the proof of Proposition 4.2, the length of all center leaves in $\mathcal{I}(V)$ are bounded from above by $\delta/2$.

Let us consider the local unstable leaf through x_0, and parameterize it such that the first parameter point is in CS_0, and the last is in CS_1. Each point in this unstable arc belongs to a unique closed curve γ_t contained in the boundary prism and consisting of two center and two stable curves.

Note that for the first parameter $t = 0$, the curve γ_0 bounds a disc, the disc CS_0. This is an open property in the set of parameters, so γ_t bound discs CS_t for small parameters $t > 0$. Let t_0 be the first parameter such that γ_{t_0} does not bound a disc. This implies that there is a center curve in $\hat{V}$ through a point in $\gamma_{t_0} \cap SU_0$ that does not reach L_1. Otherwise, by taking all center leaves in $\hat{V}$ through points in $\gamma_{t_0} \cap SU_0$, we would obtain a disc bounded by γ_{t_0}. Call D_{t_0} the set of all center leaves in $\hat{V}$ through $\gamma_{t_0} \cap SU_0$.

Now, this center leaf that does not reach L_1 is complete, but not compact, and moreover, it cannot accumulate on the boundary of the prism. So, there are uniformly sized center-stable plaques centered at points in this center leaf, contained in D_{t_0}. These center-stable plaques accumulate somewhere in the gut.

Let us take a limit center-plaque of these, and consider a foliated neighborhood of it. Then a transverse unstable arc in this foliated neighborhood meets D_{t_0} infinitely many times. But D_{t_0} is accumulated by the CS_t's. In particular, we have that the unstable arc meets infinitely many uniformly sized disjoint center-stable plaques of CS_0, which is compact. This is a contradiction. □

Theorem 4.1 implies that any non trivial invariant sub-lamination $\Lambda \subset \Gamma(f)$ without compact leaves can be extended to a foliation of M without compact leaves. Indeed, any complementary region V is an I-bundle, and hence it is diffeomorphic to the product of a boundary leaf times the open interval: $L_0 \times (0,1)$. The foliation $F_t = L_0 \times \{t\}$ induces a foliation of V.

This has the following consequence in case the fundamental group of M is nilpotent:

Proposition 4.5. *If M is a compact 3-manifold with nilpotent fundamental group, and $\emptyset \subsetneq \Lambda \subsetneq M$, is an invariant sub-lamination of $\Gamma(f)$, then there exists a leaf of Λ that is a periodic 2-torus with Anosov dynamics.*

Proof. If Λ has a compact leaf, let us consider the set Λ_c of all compact leaves of Λ. Λ_c is in fact an invariant sub-lamination, due to Theorem 2.2. Hence Proposition 3.4 implies that the boundary leaves of Λ_c are periodic 2-tori with Anosov dynamics, and we obtain the claim.

If, on the contrary, Λ does not have compact leaves, then due to Theorem 4.1 above, we can extend Λ to a foliation $\mathcal{F}$ of M without compact leaves. In particular, $\mathcal{F}$ is a Reebless foliation. Item (2) of Theorem 2.1 implies that for all boundary leaves L of Λ, $\pi_1(L)$ injects in $\pi_1(M)$, henceforth it is nilpotent.

Now, this implies that the boundary leaves can only be planes or cylinders. Theorem 3.4 implies that stable and unstable leaves of periodic points are dense in those leaves, which is impossible for the case of the plane or the cylinder. Therefore, Λ must contain a compact leaf, and due to what was shown above, it must contain a periodic 2-torus with Anosov dynamics. □

In fact, in Section 7, Corollary 7.3 we shall see that periodic 2-tori with Anosov dynamics are not possible in 3-manifolds with nilpotent fundamental group, unless the manifold is $\mathbb{T}^3$. Hence the hypotheses of Proposition 4.5 are not fulfilled, unless the manifold is $\mathbb{T}^3$. This will eliminate case (2) mentioned at the beginning of this Section.

5. Invariant tori

In this section we will prove Theorem 1.6. This theorem and the results in this section are valid for any 3-manifold M, and do not require that its fundamental group be nilpotent. Moreover, Theorem 5.1 does not even require the existence of a partially hyperbolic diffeomorphism.

Let T be an embedded 2-torus in M. We shall call T an *Anosov torus* if there exists a homeomorphism $g: M \to M$ such that T is g-invariant, and $g|_T$ is homotopic to an Anosov diffeomorphism.

Also, let S be a two-sided embedded closed surface of M^3 other than the sphere. S is *incompressible* if and only if the homomorphism induced by the inclusion map $i_\#: \pi_1(S) \hookrightarrow \pi_1(M)$ is injective; or, equivalently, after the Loop Theorem, if there is no embedded disc $D^2 \subset M$ such that $D \cap S = \partial D$ and $\partial D \nsim 0$ in S (see, for instance, [12]).

The following theorem is general, and does not depend on the existence of a partially hyperbolic dynamics in the manifold.

Theorem 5.1. *Anosov tori are incompressible.*

Proof. Let T be an Anosov torus, and let us assume by contradiction that there is an embedded disk $D^2 \subset M$ such that $D \cap T = \partial D$ and $\partial D \nsim 0$ in T. Then, by splitting M along T we obtain a manifold with boundary $\overline{M}$ such that $\partial\overline{M} = T_1 \cup T_2$ where T_i, $i = 1, 2$, are two tori and at least one of them, say T_1, verifies that the homomorphism induced by the inclusion $i_\#: \pi_1(T_1) \hookrightarrow \pi_1(\overline{M})$ is not injective.

Let $g: M \to M$ be a homeomorphism such that T is g-invariant, and g is homotopic to Anosov when restricted to T. The homeomorphism g naturally induces a homeomorphism $\overline{g}: \overline{M} \to \overline{M}$ fixing T_1 (take g^2 if necessary) and such that $(\overline{g}|_{T_1})_\#: \pi_1(T_1) \to \pi_1(T_1)$ is a hyperbolic linear automorphism. Moreover, $(\overline{g}|_{T_1})_\#$ leaves $\ker(i_\#)$ invariant.

Now, since the eigenspaces of $(\overline{g}|_{T_1})_\#$ have irrational slope, it is not difficult to find $(j, 0), (0, k) \in \ker(i_\#)$ such that $j, k \in \mathbb{N} \setminus \{0\}$. Let α, β be two simple closed curves in T_1 such that α and β meet only at a single point and α^j is in the class $(j, 0)$ and β^k in the class $(0, k)$. If we delete all $\partial\overline{M}$ but a small tubular neighborhood of α, the Loop Theorem gives us a disc D_α embedded in $\overline{M}$ with boundary α and in the same way we obtain D_β with boundary β (this implies $(1, 0), (0, 1) \in \ker(i_\#)$ and, obviously, $\ker(i_\#) = \pi_1(T_1)$) Since we can assume that D_α and D_β are transverse, this leads to a contradiction with the fact that the intersection between α and β consists of one point. □

We are now in position to prove Theorem 1.6 of Page 4:

Proof of Theorem 1.6. If $\Gamma(f) = M$ then there are no Reeb components. Indeed, since f is conservative, if there were a Reeb component, then its boundary torus should be periodic. We get a contradiction from Theorem 5.1. This gives case (3).

Let us assume that $\Gamma(f) \neq M$. If $\Gamma(f)$ contains a compact leaf then the set of compact leaves is a sub-lamination Λ of $\Gamma(f)$ by Theorem 2.2. Proposition 3.4 implies that the boundary leaves of Λ are Anosov tori, and we obtain case (1).

If $\Gamma(f) \neq M$ and contains no compact leaves, then Theorem 4.1 and Proposition 3.4 give us case (2). $\square$

6. Growth of curves

In this section we will show that the diameter of an unstable curve in $\tilde{M}$ grows exponentially under the action of f. Then, following the ideas in [1], we will use this fact in Section 7 to prove that the action of the homomorphism f_* induced by f on the first homology group is hyperbolic.

The main result in this section is the following:

Proposition 6.1. *If $f : M \to M$ is a partially hyperbolic diffeomorphism of a 3-nilmanifold, such that either $E^c \oplus E^u$ or $E^s \oplus E^u$ is tangent to an invariant foliation $\mathcal{F}$, then there is a constant $C > 0$ such that* $\text{length}(u) \leq C\,(\text{diam}(u))^4$ *for all unstable arcs u in the universal covering $\tilde{M}$ of M.*

In particular, the diameter of an unstable curve in $\tilde{M}$ grows exponentially fast under the action of $\tilde{f}$.

Proposition 6.1 will follow from Proposition 6.2 below and Lemma 6.4. Let us introduce some definitions first.

Given a compact manifold M and x in $\tilde{M}$, the universal covering of M, let us define the function $\mathsf{v} : \mathbb{R}^+ \to \mathbb{R}^+$ in the following way: for any $x \in \tilde{M}$ let $\mathsf{v}_x(r) = vol\,(B(x,r))$, where $B(x,r)$ is the ball centered at x of radius r. Notice that there is $K > 0$ such that $\mathsf{v}_x(r) \leq K\mathsf{v}_y(r)$ for any two points x and y in $\tilde{M}$. So let us fix $x_0 \in \tilde{M}$ and call $\mathsf{v}(r) = \mathsf{v}_{x_0}(r)$. The following proposition holds for a general 3-manifold M.

Proposition 6.2. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism of a 3-manifold. Assume that either $E^s \oplus E^u$ or $E^c \oplus E^u$ is tangent to an invariant foliation $\mathcal{F}$. Then there is a constant $C > 0$ such that if $u \subset \tilde{M}$ is an unstable arc then* $\text{length}(u) \leq C\mathsf{v}\,(\text{diam}(u))$.

Proof. In the first place, we have that $\mathcal{F}$ is a Reebless foliation. This is a consequence of Theorem 3.5 in the case of a foliation tangent to $E^c \oplus E^u$, and of Theorem 5.1 in case of a foliation tangent to $E^s \oplus E^u$, because the boundary of a Reeb component must be periodic.

Let us prove the following lemma:

Lemma 6.3. *If $\tilde{\mathcal{F}}$ is the lift to the universal cover of a transversely orientable Reebless foliation, then there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \cap \tilde{L}$ is connected for all leaves $\tilde{L}$ of $\tilde{\mathcal{F}}$ for all $x \in \tilde{M}$.*

Proof of Lemma. Otherwise we would find a product neighborhood of x such that there are two disjoint plaques of $\tilde{L}$ met by the same short transverse arc s. Recall that $\mathcal{F}$ is transversely orientable. Let us consider an arc $\alpha \subset \tilde{L}$ joining these two intersections of s and $\tilde{L}$. Cover α with foliated neighborhoods. It is easy to find an arc β close to α, such that β is transverse to $\tilde{\mathcal{F}}$, and $\beta \cup s$ contains a close curve $\tilde{\gamma}$ transverse to $\tilde{\mathcal{F}}$. This would produce a curve γ in M, which is homotopically trivial and transverse to $\mathcal{F}$. Theorem 2.1 would imply a Reeb component. $\square$

There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there is $\delta > 0$ for which:

$$\text{length}(u) > \delta \quad \Rightarrow dist(endpoint_1(u), endpoint_2(u)) > \varepsilon$$

Indeed, Lemma 6.3 precludes the possibility that the endpoints of u be in close disjoint plaques. Now, the endpoints of a long unstable arc cannot be in the same plaque of $\tilde{L}$, since $\tilde{L}$ (which is a plane, due to Theorem 2.3) is foliated by curves tangent to E^u. If the ends were in the same plaque, then there would be a short transverse curve enclosing a region. A Poincaré-Bendixon type argument would produce a contradiction.

So, given an unstable arc u there exist at least length$(u)/2\delta$ disjoint 3-balls of radius $\varepsilon/2$ with center at a point of u. Clearly, the union of these balls is contained in $B(x, \text{diam}(u))$ for some x which implies:

$$\text{length}(u) \leq \frac{2\delta}{\min\{\mathsf{v}_y(\varepsilon/2); y \in \tilde{M}\}} \mathsf{v}_x(\text{diam}(u)) \leq K\mathsf{v}(\text{diam}(u)).$$

$\square$

Lemma 6.4. *Three dimensional nilmanifolds have polynomial growth of volume. More precisely,*

$$\mathsf{v}(r) \leq Kr^4$$

Proof. Let M be a three-nilmanifold, and for each $r > 1$, consider the isomorphism L_r of $\tilde{M}$

$$L_r(x,t) = (rx, r^2 t)$$

for all (x,t) in $\tilde{M}$. See Section 7 for details of automorphisms of nilmanifolds. Then, $\hat{L}_r = D_0 L_r$ is induced by the matrix

$$\hat{L}_r = \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^2 \end{pmatrix}$$

Now, we have,

$$\mathsf{v}(r) \leq vol(L_r B(0,1)) = \int_{L_r B(0,r)} dx = \det(\hat{L}_r)\mathsf{v}(1) \leq \mathsf{v}(1) r^4$$

$\square$

7. Nilmanifolds in dimension 3

In this section we give a more detailed description of three dimensional nilmanifolds. This is motivated by the fact that the main theorem in this paper, Theorem 1.3, applies only to nilmanifolds, due to Theorem 1.4.

Also, in this section we will see that $\Gamma(f)$ cannot have a periodic compact leaf. This will rule out the possibility (2) mentioned at the beginning of Section 4, and will impose some restrictions to the possibility (3).

Let $\mathcal{H}$ be the group of upper triangular 3×3 matrices with ones in the diagonal. This is a non-abelian nilpotent simply connected three dimensional Lie group. We may identify $\mathcal{H}$ with the pairs (x,t) where $x=(x_1,x_2)\in\mathbb{R}^2$, $t\in\mathbb{R}$,

$$(x,t)\cdot(y,s)=(x+y,t+s+x_1y_2) \qquad \text{and} \qquad (x,t)^{-1}=(-x,x_1x_2-t).$$

We have the projection $p:\mathcal{H}\to\mathbb{R}^2$, $p(x,t)=x$ which is also an homomorphism.

If we denote by $\mathfrak{h}$ the Lie algebra of $\mathcal{H}$, then we may also identify $\mathfrak{h}$ with the pairs (x,t) where $x=(x_1,x_2)\in\mathbb{R}^2$, $t\in\mathbb{R}$. We have the exponential map $\exp:\mathfrak{h}\to\mathcal{H}$ given by $\exp(x,t)=(x,t+\frac{1}{2}x_1x_2)$, exp is one to one and onto; and its inverse, the logarithm, $\log:\mathcal{H}\to\mathfrak{h}$ is given by $\log(x,t)=(x,t-\frac{1}{2}x_1x_2)$.

The homomorphisms from $\mathcal{H}$ to $\mathcal{H}$ are of the form $L(x,t)=(Ax,l(x,t))$, where

$$A=\begin{pmatrix} a & b\\ c & d\end{pmatrix} \qquad \text{and} \qquad l(x,y)=\alpha x_1+\beta x_2+\det(A)t+\frac{ac}{2}x_1^2+\frac{bd}{2}x_2^2+bcx_1x_2.$$

Let $\hat{L}=D_0L$. Then

$$\hat{L}=\begin{pmatrix} a & b & 0\\ c & d & 0\\ \alpha & \beta & \det(A)\end{pmatrix}$$

and $\exp\left(\hat{L}(x,t)\right)=L\left(\exp(x,t)\right)$.

The centralizer of $\mathcal{H}$ is exactly $\mathcal{H}_1=[\mathcal{H},\mathcal{H}]$ which consists of the elements of the form $(0,t)$. Any homomorphism from $\mathcal{H}$ to $\mathcal{H}$ must leave $\mathcal{H}_1$ invariant.

Any lattice in $\mathcal{H}$ is isomorphic to $\Gamma_k=\{(x,t):x\in\mathbb{Z}^2,t\in\frac{1}{k}\mathbb{Z}\}$, for k a positive integer.

The automorphisms of $\mathcal{H}$ are exactly the ones with $\det(A)\neq 0$ and the automorphisms leaving Γ_k invariant are the ones with $A\in GL(2,\mathbb{Z})$ (the matrices with integer entries and determinant ± 1) and $\alpha,\beta\in\frac{1}{k}\mathbb{Z}$. On the other hand, every automorphism of Γ_k extends to an automorphism of $\mathcal{H}$.

Lemma 7.1. *If S is a subgroup of Γ_k isomorphic to $\mathbb{Z}^2$, then $S\cap\mathcal{H}_1\neq\{(0,0)\}$.*

Proof. Let (x,t) and (y,s) generate S. Then, $(x,t)\cdot(y,s)=(y,s)\cdot(x,t)$ implies $x_1y_2=x_2y_1$. So, $y=\frac{p}{q}x$. Now, $(x,t)^p\cdot(y,s)^{-q}=(0,u)$ for some $u\in\mathbb{R}$. The fact that (x,t) and (y,s) generate S implies $u\neq 0$, so $(0,u)\in\mathcal{H}_1\cap S$ □

We define the quotient $N_k = \mathcal{H}/\Gamma_k$ given by the relation $(x,t) \sim (y,s)$ iff $(x,t)^{-1} \cdot (y,s) \in \Gamma_k$. A *compact 3-nilmanifold* is defined as either N_k for some $k \in \mathbb{N}$, or $\mathbb{T}^3$. The first homotopy group is $\pi_1(N_k) = \Gamma_k$ and two endomorphisms of N_k are homotopic if and only if their actions on Γ_k coincide. Moreover, any endomorphism of N_k is homotopic to an automorphism as described above leaving Γ_k invariant.

Given $f : N_k \to N_k$, with induced automorphism on Γ_k, $f_\# = L$, if $F : \mathcal{H} \to \mathcal{H}$ is a lift of f to $\mathcal{H}$, then $F(zn) = F(z)L(n)$ for every $n \in \Gamma_k$, $z \in \mathcal{H}$. Moreover, $F(z) = \xi(z)L(z)$, where $\xi : \mathcal{H} \to \mathcal{H}$ is such that $\xi(zn) = \xi(z)$ for every $n \in \Gamma_k$, $z \in \mathcal{H}$ and for $k > 0$, $F^k(z) = \xi_k(z)L^k(z)$ where

$$\xi_{k+1}(z) = \xi\left(F^k(z)\right) L\left(\xi\left(F^{k-1}(z)\right)\right) \dots L^k\left(\xi(z)\right) L^{k+1}(z) \tag{7.2}$$

The projection $p : N_k \to \mathbb{T}^2$, $p\left((x,t)\cdot\Gamma_k\right) = x + \mathbb{Z}^2$ induces an isomorphism $p_* : H_1(N_k) \to H_1(\mathbb{T}^2)$. Let d be a right invariant metric on $\mathcal{H}$, for instance $d\left((x,t),0\right) = |x| + |t - \frac{1}{2}x_1x_2|$ where $x = (x_1,x_2)$. The exponential $\exp : \mathfrak{h} \to \mathcal{H}$ is an isometry if the metric in $\mathfrak{h}$ is $|(x,t)| = |x| + |t|$.

We have the following proposition:

Proposition 7.2. *Let $f : N_k \to N_k$ be a partially hyperbolic diffeomorphism. If f does not have the accessibility property then $E^s \oplus E^u$ integrates to a foliation such that any f-invariant sub-lamination is the whole manifold M. Moreover, the action of $f_* = A$ on $H_1(N_k,\mathbb{Z})$ is hyperbolic and hence there is a semiconjugacy $h : N_k \to \mathbb{T}^2$ homotopic to p such that $h \circ f = Ah$.*

Proof. Proposition 4.5 and Theorem 5.1 imply that either $E^s \oplus E^u$ integrates to a foliation having the minimality property stated above, or there is a torus T invariant by f^k whose homotopy group injects in $\pi_1(N_k)$ and such that $f^k_\# | \pi_1(T)$ is hyperbolic.

If this were the case, since $\pi_1(T) \sim \mathbb{Z}^2$, using Lemma 7.1 we get that $\pi_1(T) \cap \mathcal{H}_1 \neq \{0\}$. On the other hand, for any automorphism L of Γ_k, $L(x,t) = (x,t)^{\pm 1}$ for every $(x,t) \in \mathcal{H}_1$ which gives a contradiction.

So we have that $E^s \oplus E^u$ integrates to a foliation. Let us call $L = f_\#$. Notice that $L(x,t) = (Ax, l(x,t))$ where $A = f_*$ the action of f on $H_1(N_k,\mathbb{Z})$. Thus, if A is not hyperbolic, then it is not hard to see that for $k > 0$, $d(L^k z, 0) \leq Ck^2 d(z,0)$ for some constant $C > 0$. Take $F(z) = \xi(z)Lz$ a lift of f to $\mathcal{H}$. Then, using formula 7.2 and that d is a right invariant metric, we have that for every $z \in \mathcal{H}$,

$$\begin{aligned}
d\left(F^k(z),0\right) &\leq \sum_{i=0}^{k-1} d\left(L^i \xi\left(f^{k-1-i}(z)\right),0\right) + d\left(L^k(z),0\right) \\
&\leq d\left(\xi\left(f^{k-1}(z)\right),0\right) + \sum_{i=1}^{k-1} Ci^2 d\left(\xi\left(f^{k-1-i}(z)\right),0\right) + Ck^2 d\left(z,0\right) \\
&\leq Ck^3 + Ck^2 d(z,0)
\end{aligned}$$

since $d(\xi(z), 0) \leq C$ for every $z \in \mathcal{H}$.

Thus, if u is an unstable arc in $\mathcal{H}$, then for $k > 0$, $diam\left(F^k(u)\right) \leq C(u)k^3$ for some constant $C(u)$ that does not depend on k. On the other hand, by Proposition 6.1, the growth of $F^k(u)$ should be exponential. Thus we get a contradiction and hence A must be hyperbolic. The existence of h follows exactly as in [8]. $\square$

Let us note that we have also shown the following general Corollary (it does not require partial hyperbolicity):

Corollary 7.3. *If M is a compact 3-nilmanifold, and $T \subset M$ is an Anosov torus, then $M = \mathbb{T}^3$.*

Let us observe that alternatively we can view N_k as the mapping torus of the toral automorphism $A_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ in the following way. Let us define

$$\Psi_k : \mathcal{H} \to \mathbb{R}^3 \quad \text{so that} \quad \Psi_k \begin{pmatrix} 1 & x_1 & t \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} = (-kt, x_1, x_2).$$

It is easy to see that there exists $\psi_k : N_k \to M_k$, where M_k is the mapping torus of A_k, making the following diagram commute:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\Psi_k} & \mathbb{R}^3 \\ \approx\downarrow & & \downarrow\sim \\ N_k & \xrightarrow{\psi_k} & M_k \end{array}$$

Here $\approx$ is the relation induced by Γ_k, and $\sim$ is the relation induced by A_k in the mapping torus, namely $(x, 1) \sim (A_k x, 0)$.

Moreover, ψ_k is a diffeomorphism. In the next section we shall view the nilmanifolds N_k as the mapping tori M_k, and use them indistinctively.

8. Proof of Theorem 1.3

Let $f : M \to M$ be a conservative partially hyperbolic diffeomorphism of a compact orientable three dimensional nilmanifold. As it was shown in Proposition 7.2, if f does not have the accessibility property, then $E^s \oplus E^u$ integrates to a foliation $\mathcal{F}^{su}$ with the following minimality property: any closed, nonempty, f-invariant set saturated by leaves is the whole manifold M. In this last section we shall prove that the existence of such a foliation leads us to a contradiction. Without loss of generality we may assume, by taking a double covering if necessary, that $\mathcal{F}^{su}$ is transversely orientable. Observe that the double covering of a nilmanifold is again a nilmanifold.

Lemma 8.1. *If $\mathcal{F}^{su}$ has a compact leaf, then $M = \mathbb{T}^3$.*

Proof. Observe that there are no periodic compact leaves due to the minimality property of the foliation. On the other hand, if there is a compact non-invariant leaf T (it must be a torus) then $\overline{\{f^n(T); n \in \mathbb{Z}\}} = M$. This implies that all the leaves are tori, due to Theorem 2.2. Then M is the mapping torus of a linear automorphism of $\mathbb{T}^2$ that commutes with a hyperbolic one. Indeed, M is a torus bundle over the circle, whose fibers are the tori of the foliation. Let us cut M along one of these tori T. We obtain a manifold with boundary diffeomorphic to $T \times [0, 1]$. T has two transverse foliations with irrational slope. In order to reobtain M we identify $T \times \{0\}$ with $T \times \{1\}$ by means of a diffeomorphism ψ of T which preserves the two transverse foliations. ψ is isotopic to a linear automorphism A of T. A preserves the asymptotic directions of both foliations. Hence, A is either hyperbolic or $\pm id$.

Now, the only orientable three dimensional nilmanifold satisfying this property is $\mathbb{T}^3$. □

We shall say that a set Λ is a ***minimal set*** of a foliation $\mathcal{F}$ if Λ is a sub-lamination of $\mathcal{F}$ such that all leaves of Λ are dense in Λ. The proof of the following theorem can be found in [13, Theorem 4.1.3.]

Theorem 8.2. *Let $\mathcal{F}$ be a codimension one C^0-foliation without compact leaves of a three dimensional compact manifold M. Then, $\mathcal{F}$ has a finite number of minimal sets.*

$\mathcal{F}$ is a ***minimal foliation*** if all leaves of $\mathcal{F}$ are dense in M.

Lemma 8.3. *$\mathcal{F}^{su}$ is a minimal foliation.*

Proof. $\mathcal{F}^{su}$ has no compact leaves. Call $K_1, \dots, K_k$ the minimal sets of $\mathcal{F}^{su}$, whose number is finite due to Theorem 8.2. $K = K_1 \cup \dots \cup K_k$ is an f-invariant sub-lamination of $\mathcal{F}^{su}$. Since two different minimal sets have empty intersection, one K_i is periodic and hence $K_i = M$ proving the minimality of $\mathcal{F}^{su}$. □

This kind of foliations is already classified in our context, as it is shown by Plante in [14]. Concretely, Theorem 4.1 plus Remark (i) of page 227 give:

Theorem 8.4 (Plante). *Let M be a compact oriented 3-manifold such that $\pi_1(M)$ is solvable, and let $\mathcal{F}$ be a transversely oriented codimension-one minimal foliation. Then $\mathcal{F}$ is homeomorphic to the foliation of the mapping torus*

$$M_\phi = \mathbb{T}^2 \times [0, 1]/(x, 1) \sim (\phi(x), 0)$$

given by $\mathcal{G} \times [0, 1]/\sim$, where $\mathcal{G}$ is a ϕ-invariant foliation of $\mathbb{T}^2$, which is covered by a parallel line foliation of the plane.

The theorem above was originally formulated for C^2 foliations. This hypothesis is necessary only to avoid Denjoy-type exceptional minimal sets. The hypothesis of the minimality of $\mathcal{F}$ avoids this kind of phenomenon, obtaining the same result for C^0 foliations.

Now, if our manifold is N_k, then ϕ in Theorem 8.4 must be isotopic to $A_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, due to the final comment of Section 7. In particular, $\mathcal{G}$ is isotopic to an A_k-invariant foliation of $\mathbb{T}^2$. The only possibility is that $\mathcal{G}$ is a foliation by circles.

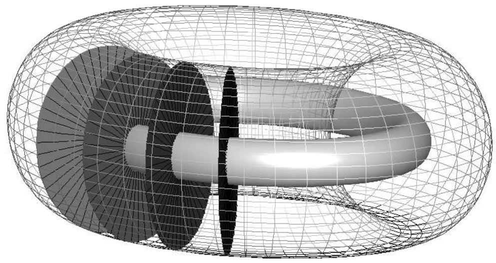

FIGURE 3. The foliation $\mathcal{F}^{su}$ of the mapping torus N_k

REMARK 8.5. *Moreover, $\mathcal{G}$ is a foliation by horizontal circles, that is, $\mathcal{F}^{su}$ is a foliation isotopic to the one appearing in Figure 3. Indeed, since $\mathcal{G}$ is isotopic to an A_k-invariant foliation of $\mathbb{T}^2$, and $\mathcal{F}^{su}$ is induced on N_k by the foliation $\mathcal{G} \times [0,1]$, then $\mathcal{F}^{su}$ is induced by the foliation of $\mathcal{H}$ having leaves of the form $(t, x_2) \mapsto \Psi_k \begin{pmatrix} 1 & x_1 & t \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}$ where x_1 is constant.*

Now, let $h : N_k \to \mathbb{T}^2$ be the semiconjugacy given by Proposition 7.2.

LEMMA 8.6. *There is $w \in N_k$ such that $h(W^\sigma(w)) = W^\sigma(h(w))$ with $\sigma = u$ or s.*

Proof. First of all we claim that there exists x such that either $h(W^s(x))$ or $h(W^u(x))$ contains more than one point. If for all $y \in N_k$ we have that $h(W^u(y)) = h(W^s(y)) = h(y) \in \mathbb{T}^2$ then $h(AC(x)) = h(x)$ (where $AC(x)$ is the accessibility class of x, see Section 3). Observe that $AC(x)$ is a leaf of $\mathcal{F}^{su}$. The minimality of $\mathcal{F}^{su}$ then implies that $AC(x)$ is dense and hence, due to continuity of h, $h(N_k) = h(x)$ contradicting the surjectivity of h.

Now, take $x \in N_k$ such that $h(W^u(x))$ is a nontrivial interval of $W^u(h(x))$ and let $z \in W^u(x)$ be such that $h(z)$ is an interior point of $h(W^u(x))$. Any point $w \in \omega(z)$ satisfies that $h(W^u(w)) = W^u(h(w))$. □

Take a point w as in the previous lemma, such that for instance $h(W^u(w)) = W^u(h(w))$, and call F the $\mathcal{F}^{su}$-leaf through w. Since F is a cylinder, there exists an injective immersion $i : \mathbb{R} \times \mathbb{S}^1 \to N_k$ such that $i(\mathbb{R} \times \mathbb{S}^1) = F$. Consider also $j = h \circ i : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{T}^2$. The previous considerations about $\mathcal{F}^{su}$ imply that $j_\# : \pi_1(\mathbb{R} \times \mathbb{S}^1) \to \pi_1(\mathbb{T}^2)$ is trivial (see Remark 8.5). Observe that $j_\# = h_\# \circ i_\# =$

$p_\# \circ i_\#$. Then there exists $\tilde{\jmath} : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}^2$ such that $j = \pi \circ \tilde{\jmath}$ where $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ is a covering projection.

Remark 8.5 also implies that $\tilde{\jmath}(\mathbb{R} \times \mathbb{S}^1)$ is contained in bounded neighborhood of $W^u(\tilde{\jmath}(i^{-1}(w)) \subset \tilde{\jmath}(\mathbb{R} \times \mathbb{S}^1)$. See Figure 4

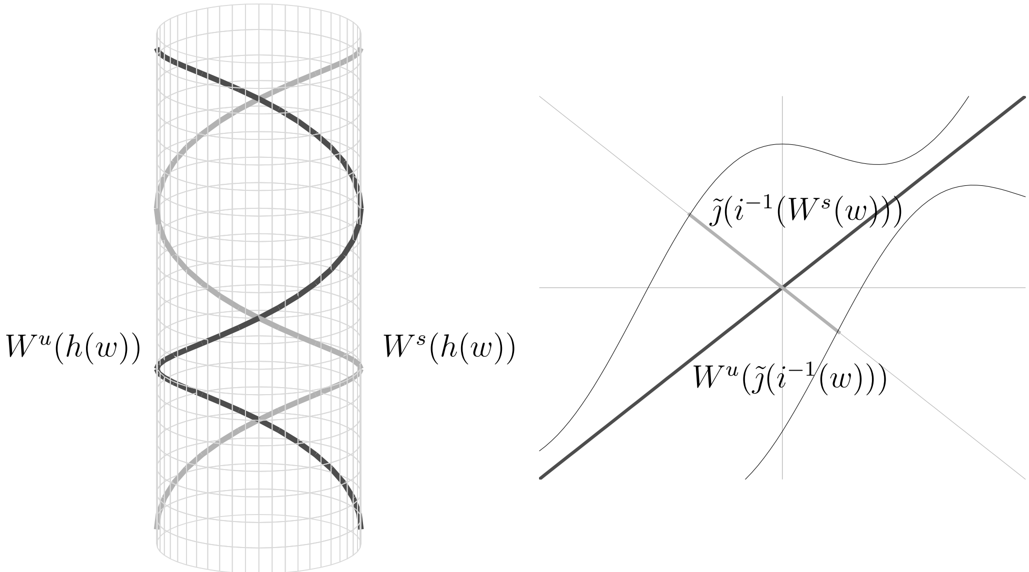

Figure 4. Proof of Theorem 1.3

Consider now $i^{-1}(W^s(w))$. Since the foliation $\mathcal{F}^s$ has neither singularities nor compact leaves, $i^{-1}(W^s(w))$ is unbounded. It is not difficult to see that this implies that $\tilde{\jmath}(i^{-1}(W^s(w)))$ is unbounded but it is at bounded distance of $W^u(\tilde{\jmath}(i^{-1}(w)))$ which is a contradiction. This proves Theorem 1.3.

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