TORI WITH HYPERBOLIC DYNAMICS IN 3-MANIFOLDS

F. RODRIGUEZ HERTZ, M. A. RODRIGUEZ HERTZ, AND R. URES

Abstract. Let M be a closed orientable irreducible 3-dimensional manifold. An embedded 2-torus $\mathbb{T}$ is an *Anosov torus* if there exists a diffeomorphism f over M for which $\mathbb{T}$ is f-invariant and $f_{\#}|_{\mathbb{T}} : \pi_1(\mathbb{T}) \to \pi_1(\mathbb{T})$ is hyperbolic. We prove that only few irreducible 3-manifolds admit Anosov tori: (1) the 3-torus $\mathbb{T}^3$, (2) the mapping torus of $-id$, and (3) the mapping tori of hyperbolic automorphisms of $\mathbb{T}^2$.

This has consequences for instance in the context of partially hyperbolic dynamics of 3-manifolds: if there is an invariant foliation $\mathcal{F}^{cu}$ tangent to the center-unstable bundle $E^c \oplus E^u$, then $\mathcal{F}^{cu}$ has no compact leaves [21]. This has led to the first example of a non-dynamically coherent partially hyperbolic diffeomorphism with one-dimensional center bundle [21].

1. Introduction

In this article we study 3-manifolds admitting an embedded torus invariant by a diffeomorphism inducing a hyperbolic automorphism in its first fundamental group. Our main result is that there are very few such manifolds. Our motivation in studying these objects come from interesting problems in partially hyperbolic dynamics.

An embedded 2-torus $\mathbb{T}$ in a 3-manifold M is an *Anosov torus* if there exists a diffeomorphism f over M such that the induced action of f over the fundamental group of $\mathbb{T}$ is hyperbolic. Our main result is the following:

Theorem 1.1. *A closed oriented irreducible 3-manifold admits an Anosov torus if and only if it is one of the following:*

(1) *the 3-torus*
(2) *the mapping torus of $-id$*
(3) *the mapping torus of a hyperbolic automorphism*

Moreover, we have the following result:

Theorem 1.2. *Let M be a compact orientable irreducible 3-manifold with non-empty boundary such that all the boundary components are incompressible 2-tori. Then M admits an Anosov torus if and only if $M = \mathbb{T}^2 \times [0,1]$.*

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The main reason to consider only irreducible 3-manifolds is that, as explained below, this work arises in the context of partially hyperbolic systems. But as an outcome of Burago, Ivanov [4] we obtain:

Theorem 1.3. *A 3-manifold that supports a partially hyperbolic diffeomorphism, ergodic or not, is always* irreducible*, that is every 2-sphere embedded in the manifold bounds a 3-ball.*

Indeed, a 3-manifold admitting a partially hyperbolic diffeomorphism has a codimension-one foliation with neither Reeb components nor spherical leaves [4]. This proves the claim since Rosenberg shows in [16] that any codimension-one foliation in a non-irreducible 3-manifold must have a Reeb component or a spherical leaf, see also [17].

On the other hand, it is easy to construct arbitrarily many different non-irreducible manifolds admitting Anosov tori. Indeed, if a manifold M supports an Anosov torus, then it is easy to see that the connected sum of M and any other 3-manifold will admit an Anosov torus. See Remark 2.6. An interesting question to solve would be the following:

Question 1.4. *Let M be an orientable non-irreducible 3-manifold. Then, the Kneser-Milnor theorem states that we can decompose M, uniquely up to diffeomorphisms, into a finite connected sum:*

$$M = M_1 \# M_2 \# \dots \# M_n$$

where each M_i is either irreducible, or a handle $\mathbb{S}^2 \times \mathbb{S}^1$. If M admits an Anosov torus, is one of the M_i necessarily one of the 3-manifolds listed in Theorem 1.1?

The technical difficulties in answering this question mainly arise from the non-uniqueness of the prime decomposition modulo isotopies.

1.1. **Partially hyperbolic diffeomorphisms.** A partially hyperbolic system is a diffeomorphism that leaves invariant three complementary bundles: E^s, on which the action of the derivative is contracting; E^u, on which it is expanding; and E^c on which the action is not as contracting as in E^s, nor as expanding as E^u. Namely, f admits a Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ such that Df and Df^{-1} are contractions, respectively, on E^s and E^u, and all unit vectors $v_\sigma \in E^\sigma$ with $\sigma = s, c, u$ satisfy:

$$\|Dfv_s\| < \|Dfv_c\| < \|Dfv_u\|$$

The bundles E^s, E^c and E^u are called, respectively, the *stable*, *center*, and *unstable* bundles. A recent survey of partially hyperbolic dynamics is for instance [18].

It was conjectured by Pugh and Shub in 1995 that conservative partially hyperbolic diffeomorphisms contain an open and dense set of ergodic systems. This conjecture was proven true for 3-manifolds by the authors [19]. Our question is: can we classify all 3-manifolds supporting non-ergodic partially hyperbolic diffeomorphisms?

Conjecture 1.5. *Only the following orientable 3-manifolds support non-ergodic partially hyperbolic diffeomorphisms:*

(1) *the 3-torus,*
(2) *the mapping torus of $-id$, or*
(3) *the mapping torus of a hyperbolic automorphism of the 2-torus.*

In [20] we proved that this conjecture is true for nilmanifolds: the only nilmanifold supporting a non-ergodic conservative partially hyperbolic diffeomorphism is the 3-torus. According to this conjecture, the only manifolds supporting non-ergodic partially hyperbolic diffeomorphisms would be exactly those enumerated in Theorem 1.1, namely, the manifolds admitting an Anosov torus.

Another problem in partially hyperbolic dynamics concerns the integrability of the center bundle E^c. Indeed, the stable and unstable bundles are known to be integrable [10], but the situation is different for the center bundle. It is an open problem to determine the conditions under which a partially hyperbolic diffeomorphism of a 3-manifold has an integrable center bundle. In [4], it is shown that there are always foliations *almost* tangent to the center bundle. Moreover, a large class of partially hyperbolic diffeomorphisms of the 3-tori have integrable center bundle, as it was recently shown in [2]. However, in [21] we give an example of a partially hyperbolic diffeomorphism of the 3-torus, having a non-integrable center bundle. This example answers a question that had been posed by many authors in the recent decades, basically since the study of partially hyperbolic systems began, see for instance [10] and [3].

This new example was inspired by the theorem below, which is one of the applications of Theorem 1.1 and gives a more accurate description of dynamically defined foliations of partially hyperbolic diffeomorphisms of 3-manifolds.

Theorem 1.6. [21] *Let M be a closed orientable 3-dimensional manifold and $f : M \to M$ be a partially hyperbolic diffeomorphism with dynamically coherent center-unstable bundle $E^c \oplus E^u$. Then, the center-unstable foliation $\mathcal{F}^{cu}$ has no compact leaves.*

Indeed, to study the integrability of the center bundle, it is standard to analyze the behavior of the so called *center-stable* and *center-unstable* bundles, that are respectively, the Whitney sums $E^s \oplus E^c$ and $E^c \oplus E^u$. One says that the center-stable bundle $E^s \oplus E^c$ is *dynamically coherent* if there exists an invariant foliation tangent to it. In this case, the tangent foliation is called the *center-stable foliation.* Analogously one defines the center-unstable foliation.

Theorem 1.6 does not prevent the existence of tori, even invariant, tangent to the center-unstable bundle, but it asserts the impossibility of the existence of such tori as part of an invariant foliation tangent to the center-unstable bundle. In

[21] we give examples of a partially hyperbolic diffeomorphism of $\mathbb{T}^3$ with center-unstable tori. The center foliation of this example in particular is not uniquely integrable although some of these examples are *dynamically coherent*, that is both the center-stable and the center-unstable bundles are dynamically coherent.

1.2. **Organization of paper.** This paper is organized as follows. In Section 2 we include preliminary concepts of 3-manifolds in order that this paper be as self-contained as possible. Proposition 2.4, which is introduced in Section 2 and proven in Section 3, splits the proof of the main theorem into three distinct cases. These cases are studied one-at-a-time, in Sections 4, 5 and 6. Then, Section 7 brings these case-by-case results together into a completed proof of Theorem 1.1, and concludes with a short proof of Theorem 1.2.

Some cases in Theorem 1.1 had already been studied by F. Waldhausen in his classification of graph manifolds [22]. Namely, he proves Theorem 1.1. when M is a graph manifold, that is, when every component in the JSJ-decomposition is Seifert. Indeed, in Lemma 5.7 of [22], Waldlhausen proves, by using coordinates, that such a manifold can not contain an Anosov torus, unless it is finitely covered by the 3-torus. Nevertheless, for the sake of completeness, we include a proof of this case.

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2. Preliminaries

We shall assume from now on, that M is an irreducible 3-manifold. We will cut M along certain tori, and will obtain certain 3-manifolds with boundary that are easier to handle.

An orientable surface $S \neq \mathbb{S}$ embedded in M is *incompressible* if the homomorphism induced by the inclusion map $i_\# : \pi_1(S) \hookrightarrow \pi_1(M)$ is injective; or, equivalently, if there is no embedded disc $D^2 \subset M$ such that $D \cap S = \partial D$ and $\partial D \nsim 0$ in S (see, for instance, [11, Page 10]).

A *Seifert manifold* is one which admits a decomposition into disjoint circles, the *fibers*, such that each fiber has a neighborhood diffeomorphic, preserving fibers, to a neighborhood of a fiber in some *model Seifert fibering* of $\mathbb{S}^1 \times D^2$. A model Seifert fibering of $\mathbb{S}^1 \times D^2$ is a decomposition of $\mathbb{S}^1 \times D^2$ into disjoint circles (fibers), constructed as follows. Starting with $[0,1] \times D^2$, decomposed into the segments $[0,1] \times \{x\}$, identify the disks $\{0\} \times D^2$ and $\{1\} \times D^2$ via a $2\pi p/q$ rotation, for $p/q \in \mathbb{Q}$, with p and q relatively prime. The segment $[0,1] \times \{0\}$ then becomes a fiber $\mathbb{S}^1 \times \{0\}$, while every other fiber in $\mathbb{S}^1 \times D^2$ is made from q segments $[0,1] \times \{x\}$. A Seifert manifold can have non-empty boundary consisting of tori. Any 3-manifold supporting a foliation by circles is Seifert (see [7])

A 3-manifold with boundary N is *atoroidal* if every incompressible torus is ∂-*parallel*, that is, isotopic to a subsurface of ∂N. A 3-manifold with boundary N

is *acylindrical* if every incompressible annulus A that is *properly embedded*, i.e. $\partial A \subset \partial N$, is ∂-parallel, by an isotopy fixing ∂A.

A closed irreducible 3-manifold admits a natural decomposition into Seifert pieces on one side, and atoroidal and acylindrical components on the other:

Theorem 2.1 (JSJ-decomposition [12], [13])**.** *If M is an irreducible closed orientable 3-manifold, then there exists a collection of disjoint incompressible tori $\mathcal{T}$ such that for each component M_i of $M \setminus \mathcal{T}$ either*

(1) *M_i is a Seifert manifold, or*
(2) *M_i is both atoroidal and acylindrical.*

Any minimal such collection is unique up to isotopy. This means, if $\mathcal{T}$ is a collection as described above, it contains a minimal sub-collection $m(\mathcal{T})$ satisfying the same claim. All collections $m(\mathcal{T})$ are isotopic.

Any minimal family of incompressible tori as described above is called a *JSJ-decomposition* of M. When it is clear from the context we shall also call a JSJ-decomposition the set of pieces obtained by cutting the manifold along these tori. Note that if M is either atoroidal or Seifert, then $\mathcal{T} = \emptyset$.

The idea of the proof of Theorem 1.1 is that, given an Anosov torus T, we can "place" T so that either T belongs to the family $\mathcal{T}$, or else T is in a Seifert component, and it is either transverse to all fibers, or it is a union of fibers of this Seifert component. See Proposition 2.4.

It is important to note the following property of Anosov tori:

Theorem 2.2. [20] *Anosov tori are incompressible.*

An Anosov torus in an atoroidal component will then be ∂-parallel to a component of its boundary. In this case, we can assume $T \in \mathcal{T}$. On the other hand, the theorem of Waldhausen below, guarantees that we can always place an incompressible torus in a Seifert manifold in a "standard" form; namely, the following: a surface is *horizontal* in a Seifert manifold if it is transverse to all fibers, and *vertical* if it is a union of fibers:

Theorem 2.3 (Waldhausen [22])**.** *Let M be a compact connected Seifert manifold, with or without boundary. Then any incompressible surface can be isotoped to be horizontal or vertical.*

The architecture of the proof of Theorem 1.1, as mentioned above, is contained in the following proposition.

Proposition 2.4. *Let T be an Anosov torus of a closed irreducible orientable manifold M. Then, there exists a diffeomorphism $f : M \to M$ and a JSJ-decomposition $\mathcal{T}$ such that*

(1) *$f|T$ is a hyperbolic toral automorphism,*

(2) *$f(\mathcal{T}) = \mathcal{T}$, and*
(3) *exactly one of the following holds*
  (a) *$T \in \mathcal{T}$*
  (b) *T is a vertical torus in a Seifert component of $M \setminus \mathcal{T}$, and T is not ∂-parallel in this component.*
  (c) *M is a Seifert manifold ($\mathcal{T} = \emptyset$), and T is a horizontal torus,*

The proposition above allows us to split the proof of Theorem 1.1 into cases. Note that case (3b) includes the case in which M is a Seifert manifold and T is a vertical torus. Before addressing the proof of Proposition 2.4, which is done in Section 3, we need the following lemma:

Lemma 2.5. *Let $\mathcal{S}$ be a Seifert fibering of a compact orientable irreducible 3-manifold M.*

(1) *If a surface with or without boundary is horizontal, it intersects all the fibers of $\mathcal{S}$.*
(2) *If $\partial M \neq \emptyset$, $\mathcal{S}$ does not admit horizontal surfaces without boundary.*

Proof. Indeed, let A be the union of all fibers which have non-empty transverse intersection with the horizontal surface S. Then A is clearly open. To see that A is closed take a sequence $x_n \to x$ such that $x_n \in A$. If the fiber of x did not intersect the horizontal surface, then there would be a neighborhood of the fiber of x not intersecting the horizontal surface. Moreover, there is a fibered neighborhood of the fiber of x not intersecting S. This neighborhood would contain the fibers of x_n, an absurdity.

To see that a horizontal manifold without boundary can live only in a manifold without boundary, assume that T is a horizontal surface without boundary and assume ∂M is not empty. Consider a fiber of $x \in \partial M$. This fiber intersects T. But then, there is a fibered neighborhood of the fiber of x diffeomorphic to a solid torus. This contradicts that x is in the boundary of M. □

Let us finish the section by explaining why we focused on irreducible manifolds.

Remark 2.6. It is easy to see that there are arbitrarily many non-irreducible 3-manifolds admitting Anosov tori. Indeed, let M be any closed 3-manifold admitting an Anosov torus T. Take p a fixed point of $f|_T$. It is easy to see that we can slightly modify f so that, if $T \times [-\epsilon, \epsilon]$ is a small tubular neighborhood of T, then $f|_{T\times\{t\}} = f|_{T\times\{0\}} = f|_T$ for all $t \in [-\epsilon, \epsilon]$.

Let us make another slight modification of f: replace $p \times \{\epsilon\} \subset T \times \{\epsilon\}$ by a small ball $B \subset M$ and take $g : M \to M$ so that $g = f$ on $M \setminus \{p \times \{\epsilon\}\}$, and g restricted to B is the identity.

If we consider now any manifold M', then there is a diffeomorphism h on $M\#M'$ such that h is f when restricted to $M \setminus B$ and h is the identity when restricted to M' minus another small ball. This implies that $M\#M'$ admits an Anosov torus. In this way we can construct arbitrarily many manifolds admitting Anosov tori.

3. Proof of Proposition 2.4

This section contains the proof of Proposition 2.4. Let M be an irreducible orientable closed 3-manifold and T be an Anosov torus. First, note that we can choose a JSJ-decomposition $\mathcal{T}$ such that T is in a Seifert piece. This is because of the so-called Enclosing Property:

Proposition 3.1 (Enclosing Property [13])**.** *There exists $\mathcal{T}$ such that either $T \in \mathcal{T}$, or else T is contained in the interior of a Seifert piece of the JSJ-decomposition generated by $\mathcal{T}$, and is not ∂-parallel in that component.*

Hence, either $T \in \mathcal{T}$ (case (3a)), or else T is in the interior of a Seifert component and it is not ∂-parallel. We want to show that in the latter case, we have either that the whole M is Seifert, and T can be put horizontally (case (3c)); or else T can be put vertically (case (3b)). Then, assume we are in the latter case.

Now, after Theorem 2.3, there is an isotopy transforming T into a horizontal or vertical torus in the interior of the Seifert component that contains it. Equivalently, there is an isotopy moving the Seifert component and fixing T, so that T is either horizontal or vertical in this new Seifert manifold. This produces a new JSJ-decomposition $\mathcal{T}'$, so that T is horizontal or vertical in the Seifert component that contains it.

Assume that T is horizontal in its Seifert component S. Then Lemma 2.5 implies that S is a closed manifold. Hence the whole manifold M is Seifert ($M = S$), and we are in case (3c). Note that in this case $\mathcal{T}' = \emptyset$.

If, on the contrary, T is vertical, recall that, by the Enclosing Property (Proposition 3.1), T is not ∂-parallel in its Seifert component, that is, T is not isotopic to any component of the boundary of the Seifert component of $M \setminus \mathcal{T}$ containing it. After the isotopy that transforms $\mathcal{T}$ into $\mathcal{T}'$, so that T is a vertical torus of the Seifert component of $M \setminus \mathcal{T}'$ that contains T, we will obviously have that T is not ∂-parallel in its Seifert component either. Hence we are in case (3b). This proves part (3) of Proposition 2.4, that is, we have obtained a JSJ-decomposition $\mathcal{T}'$.

Now, we want to obtain $f : M \to M$ satisfying items (1) and (2) of Proposition 2.4. We begin by looking for an $f : M \to M$ satisfying item (1):

Lemma 3.2. *Let T be an Anosov torus of any 3-manifold M, that is, with or without boundary, irreducible or not. Suppose that T is either in the interior of M, or $T \subset \partial M$. Then, there is a diffeomorphism $f : M \to M$ leaving T invariant, such that $f|T$ is a hyperbolic automorphism.*

Proof. First consider the case in which T is contained in the interior of M. Let g be a diffeomorphism of M leaving T invariant and such that $g|T$ is isotopic to a hyperbolic automorphism A. Consider a product neighborhood $T \times [-1,1]$ of T. Consider a differentiable map $h : T \times [-1,1] \to T$ such that, $h(t,.) = h_t$ is a diffeomorphism $\forall t \in [-1,1]$, $h_0 = A \circ g^{-1}$ and $h_t(y) = y \ \forall |t| \geq \frac{3}{4}$. Define

$$\xi(x) = \begin{cases} x & \text{if } x \notin T \times [-1,1] \\ h_t(y) & \text{if } x = (y,t) \in T \times [-1,1] \end{cases}$$

Then $f = \xi \circ g$ is the diffeomorphism we are looking for.

If M is a manifold with boundary and $T \subset \partial M$, then we can consider a neighborhood of T of the form $T \times [0,1]$. The rest of the proof follows analogously. □

In this way, we have obtained a diffeomorphism $f : M \to M$ satisfying item (1) of Proposition 2.4. In order to obtain item (2), we shall need the following version of the JSJ-decomposition:

Theorem 3.3 (Relative JSJ-decomposition, [12, 13])**.** *If M is a compact orientable irreducible 3-manifold with incompressible boundary, then there exists a family $\mathcal{T}$ of incompressible annuli and tori, such that $M \setminus \mathcal{T}$ consists of either Seifert or atoroidal and acylindrical components. Any such family $\mathcal{T}$ that is minimal by inclusion is unique up to proper isotopy.*

Let $f : M \to M$ be as in Lemma 3.2. Now cut M along T. The resulting manifold N is as in the hypotheses of Theorem 3.3, and f can be extended to this new manifold, since $f(T) = T$. Consider $f(\mathcal{T})$, the image by f of the JSJ-decomposition of M. Then $f(\mathcal{T})$ and $\mathcal{T}$ are JSJ-decompositions for N, due to the Enclosing Property (Proposition 3.1). Theorem 3.3 implies that there is an isotopy h_t fixing ∂N such that $h_0 = id$, and $h_1(\mathcal{T}) = f(\mathcal{T})$. Now $g = h_1^{-1} \circ f$ is a diffeomorphism satisfying all conditions of Proposition 2.4 for the JSJ-decomposition $\mathcal{T}$. This finishes the proof.

Observe that in our setting the relative JSJ-decomposition has no annulus. This is immediate from the fact that the manifold N was obtained from a closed one by cutting along an incompressible torus.

Remark 3.4. Note that the same idea above shows that, given a closed irreducible orientable 3-manifold M, with or without an Anosov torus, and given any diffeomorphism $f : M \to M$, a JSJ-decomposition of M, $\mathcal{T}$, can be taken so that $f(\mathcal{T}) = \mathcal{T}$. Theorem 2.1 is enough to prove this.

This result obviously holds also for compact orientable irreducible 3-manifolds with incompressible boundary.

4. Horizontal Anosov tori

We begin the proof of Theorem 1.1. Consider a closed irreducible orientable 3-manifold M, and let T be an Anosov torus. Then Proposition 2.4 states that we need only study three situations: (3a) T belonging to a JSJ-decomposition, (3b) T being a non-∂-parallel vertical torus in a Seifert component, or (3c) T being a horizontal torus in a closed Seifert manifold. In this section we study case (3c). The conclusion is that there are only two manifolds admitting this situation:

Proposition 4.1. *Let M be a closed orientable irreducible Seifert manifold that supports a horizontal Anosov torus. Then M is either $\mathbb{T}^3$ or the mapping torus of $-id$ on $\mathbb{T}^2$.*

The rest of this section is devoted to proving this proposition. Let M be a closed orientable irreducible Seifert manifold, and let T be a horizontal torus in M. In [11, Page 30] we can see that only six such manifolds admit horizontal tori:

(1) $M_1 = \mathbb{T}^3$,
(2) M_2 is the mapping torus of $-id$ on $\mathbb{T}^2$, that is, $\mathbb{S}^1 \tilde{\times} \mathbb{S}^1 \tilde{\times} \mathbb{S}^1$
(3) M_3 is the mapping torus of $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$
(4) M_4 is the mapping torus of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
(5) M_5 is the mapping torus of $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$
(6) $M_6 = N \cup_\varphi N$

In the last case, $N = \mathbb{S}^1 \tilde{\times} \mathbb{S}^1 \tilde{\times} [0,1]$ is the twisted I-bundle over the Klein bottle. N can be obtained in the following way: Let $\rho : \mathbb{T}^2 \to \mathbb{T}^2$ be an involution (without fixed points) such that the identification of x with $\rho(x)$ gives the Klein bottle. We get N from $\mathbb{T}^2 \times [-1,1]$ by identifying (x,t) with $(\rho(x), -t)$. If $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ take $\rho(x,y) = (x + \frac{1}{2}, -y)$.

M_6 is the closed manifold formed by two copies of N, glued together along its boundary. ∂N is a 2-torus and the two copies of ∂N are glued together by the automorphism $\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. M_6 is foliated by tori with the exception of two fibers that are Klein bottles. Since $(\varphi \circ \rho)^4$ is the identity map we get that the interval fibers of the two copies of N glue together to form a Seifert fibering of M_6.

The first five manifolds are torus bundles over $\mathbb{S}^1$ and M_6 is a fibration by tori except for two fibers. For all the six manifolds the horizontal torus is isotopic to a fiber. We shall then assume that T is a fiber of the M_i.

Let f be the hyperbolic automorphism of Proposition 2.4. For M_i with $i = 1, \dots, 5$, the manifolds are mapping tori for some automorphism h_i. Since T is a fiber, h_i commutes with f. But only id and $-id$ commute with a hyperbolic automorphism. This implies that M_3, M_4 and M_5 do not admit horizontal Anosov tori.

Finally, we will show that M_6 does not admit a horizontal Anosov torus. Indeed, if the manifold is M_6, the horizontal Anosov torus T splits the manifold into two components diffeomorphic to N. We have that $\partial N = T$. The following general lemma precludes the possibility that M_6 admit a horizontal Anosov torus, and finishes the proof of Proposition 4.1:

Lemma 4.2. *If N is a compact orientable 3-manifold such that ∂N is a torus T, then T is not an Anosov torus.*

In order to prove Lemma 4.2, we shall need the following:

Lemma 4.3. [11, Lemma 3.5] *Let M be a compact orientable 3-manifold with boundary ∂M. Consider the inclusion $i_* : H_1(\partial M) \hookrightarrow H_1(M)$. Let* $\ker(i_*)$ *be the kernel of the map induced by the inclusion i_*, and let* $\operatorname{rank}(H_1(\partial M))$ *be the rank of $H_1(\partial M)$. Then*

$$\operatorname{rank}(\ker(i_*)) = \frac{1}{2}\operatorname{rank}(H_1(\partial M)).$$

Here "rank" means the number of $\mathbb{Z}$ summands in a direct sum splitting into cyclic groups. If the homology with coefficients in $\mathbb{Q}$ is used, rank can be replaced by "dimension".

In fact, Lemma 3.5 of [11] states that

$$\operatorname{rank}(\operatorname{im}(\partial)) = \frac{1}{2}\operatorname{rank}(H_1(\partial M)) \tag{4.1}$$

where $\operatorname{im}(\partial)$ stands for the image of the boundary map $\partial : H_2(M, \partial M) \to H_1(\partial M)$. The fact that the following sequence

$$H_2(M, \partial M) \xrightarrow{\partial} H_1(\partial M) \xrightarrow{i_*} H_1(M)$$

is exact implies that $\operatorname{im}(\partial)$ is isomorphic to $\ker(i_*)$, hence $\operatorname{rank}(\operatorname{im}(\partial)) = \operatorname{rank}(\ker(i_*))$.

To prove Lemma 4.2, consider a compact orientable 3-manifold N such that ∂N is a torus. Then $\operatorname{rank}(H_1(\partial N)) = 2$. Lemma 4.3 implies that the rank of the kernel of the inclusion $i_* : H_1(T) \to H_1(M)$ is one. This implies that $K = \ker(i_*)$ is a one-dimensional subspace of $H_1(T)$. We shall have that $f_*(K) = K$, where $f_* : H_1(T) \to H_1(T)$ is the isomorphism induced by any diffeomorphism $f : N \to N$. This implies that f_* has an eigenvalue which is ± 1. Hence f cannot be isotopic to a hyperbolic automorphism on T. This implies that T cannot be an Anosov torus. This finishes the proof of Lemma 4.2, and hence of Proposition 4.1.

5. Vertical Anosov tori

Continuing with the proof of Theorem 1.1, we consider an Anosov torus T of an irreducible orientable closed 3-manifold M, and study now the situation (3b) of Proposition 2.4; namely, T is a vertical torus in the interior of a Seifert component of the JSJ-decomposition of M, that is not ∂-parallel.

The main result of this section is that in this case, M is like in the case (3c): either M is $\mathbb{T}^3$ or M is the mapping torus of $-id$ on $\mathbb{T}^2$. More precisely:

Proposition 5.1. *Let M be a compact connected orientable irreducible Seifert manifold, with or without boundary, admitting a vertical Anosov torus T. Then there are only three possibilities:*

(1) $M = T \times [-1, 1]$,
(2) $M = \mathbb{T}^3$, *or*
(3) M *is the mapping torus of* $-id$ *on* $\mathbb{T}^2$.

If M is a Seifert manifold with or without boundary and T is a vertical Anosov torus in M, then we can split M by cutting it along T, and we obtain a Seifert manifold with an Anosov torus as one of the boundary components. Hence we can always consider that M is a Seifert manifold with $T \subset \partial M$. The proof of Proposition 5.1 is then reduced to the proof of:

Proposition 5.2. *Let N be a compact connected orientable irreducible Seifert manifold with an Anosov torus $T \subset \partial N$. Then, $N = T \times [0, 1]$.*

Indeed, if M is a manifold as in the hypothesis of Proposition 5.1, and we split M along the vertical Anosov torus T, then each component of $M \setminus T$ is a manifold N in the hypotheses of Proposition 5.2, and hence each component N is of the form $\mathbb{T}^2 \times [0, 1]$. If $M \setminus T$ has two components, this readily implies that $M = T \times [-1, 1]$. Otherwise, the manifold N obtained by splitting M along T is connected and is $T \times [0, 1]$. Then M is a mapping torus of an automorphism h of $T = \mathbb{T}^2$.

But, since T is an Anosov torus, there is a diffeomorphism $f : N \to N$ such that $f|T = f|\partial N$ is a hyperbolic toral automorphism, see Lemma 3.2. Now, h has to commute with f on T. The only possibilities for h are then: $h = id$, $h = -id$ or h is a hyperbolic automorphism of $\mathbb{T}^2$. The last possibility corresponds to a mapping torus that is not a Seifert manifold. Hence, we can only have that M is the mapping torus of $\pm id$ on $\mathbb{T}^2$, as claimed.

To finish the proof of Proposition 5.2, it is convenient to recall that most Seifert manifolds have a unique Seifert fibration up to isotopy. Namely, we have the following:

Lemma 5.3. [11, Lemma 1.15] *The only compact connected* orientable *Seifert manifolds with boundary, admitting two Seifert fibrations that are non-isotopic in their boundary are the following:*

(1) *the solid torus* $\mathbb{D}^2 \times \mathbb{S}^1$
(2) *the twisted I-bundle over the Klein bottle* $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^1 \widetilde{\times} [0, 1]$*, or*
(3) *the torus cross the interval* $\mathbb{T}^2 \times [0, 1]$

Now, let $T \subset \partial N$ be an Anosov torus of N as in Proposition 5.2. Then there are two Seifert fibrations of N that are not isotopic on T. Indeed, take any Seifert fibration of N, and consider the diffeomorphism $f : N \to N$ such that $f|T$ is a hyperbolic automorphism (Lemma 3.2). The fibration of N restricted to T is not isotopic to its f-image (another Seifert fibration) on T. Then Lemma 5.3 implies that N is either the solid torus, the twisted I-bundle over the Klein bundle, or the torus cross the interval.

If N is either the solid torus, or the twisted I-bundle over the Klein bottle then, N has a connected boundary consisting exactly of one torus, that is, $\partial M = \mathbb{T}^2$. Lemma 4.2 implies that T cannot be the boundary of N. The only possibility left is that $M = T \times [0, 1]$. This finishes the proof of Proposition 5.2, and hence of Proposition 5.1.

6. Anosov tori in the JSJ-decomposition

In this section, we consider case (3a) of Proposition 2.4, where the Anosov torus T is one of the tori of the JSJ-decomposition. Note that T can be either the boundary of a Seifert component, in which case the component is $T \times [0, 1]$, as we proved in Proposition 5.2; or else T is the boundary of an atoroidal and acylindrical component.

The main result of this section is that an Anosov torus T is never, in fact, a boundary of an atoroidal and acylindrical component of a JSJ-decomposition. This is the most delicate part of the proof of Theorem 1.1. With this result it is easy to finish the proof of Theorem 1.1, as it is seen in Section 7.

Proposition 6.1. *Let M be a compact, connected, orientable, irreducible, atoroidal and acylindrical 3-manifold such that ∂M consists of incompressible tori. Then no component of ∂M is an Anosov torus.*

The strategy is to assume that there is an Anosov torus $T \subset \partial M$, and then use its properties to build an incompressible annulus that is not ∂-parallel.

Claim 1. *For each torus $T \subset \partial M$, there exists a compact connected orientable incompressible properly embedded surface $S \subset M$, such that ∂S contains an essential curve γ in T and S is not a ∂-parallel annulus. By essential we mean non-nullhomotopic.*

By Lemma 4.2 we have that $T \subsetneq \partial M$. To obtain S we construct a manifold N such that $\partial N = T \sqcup T$: take two copies of M, $M \sqcup M$ and glue all corresponding pairs of connected components of boundaries of M, except $T \sqcup T$. In this way we obtain a connected compact orientable 3-manifold N such that $\partial N = T \sqcup T$. Note that $H_1(\partial N) \neq 0$, so proceeding as in the previous case, we obtain an incompressible compact connected orientable properly embedded surface S representing a non-trivial homology class in $H_2(N, \partial N)$ such that ∂S is non-trivial in $H_1(\partial M)$.

Without loss of generality, we may assume that S is transverse to ∂M. Cutting along ∂M we obtain a new surface $R = S \cap M$, possibly non-incompressible and non-connected, whose boundary is non-trivial in $H_1(\partial M)$, since $R \sqcup (R \setminus S) = S$. Let us see that we can cut R along a finite number of curves so that R becomes incompressible.

Let R_i be a component of R and suppose $\pi_1(R_i) \hookrightarrow \pi_1(M)$ is not injective. Then there is a disk D realizing this non-injectivity [11, Corollary 3.3], that is, there is a disc D such that $\partial D = D \cap R_i$ is a non-null homotopic circle in R_i. If we cut R_i along ∂D we obtain a new surface that is in the same homology class, since the cutting curve and D are now duplicated in the boundary of R_i, but counted with different signs. Note that this new R_i is also properly embedded.

Since R is compact, this surgery simplifies R_i. Indeed, either ∂D separates R_i, in which case the surgery splits R_i into two components of lower genus; or not, in which case the surgery reduces the genus of R_i. We can perform finitely many cuttings until each resulting surface R_i satisfies that $\pi_1(R_i) \hookrightarrow \pi_1(M)$ is injective, so R_i is incompressible.

Note that this procedure did not change ∂R. So, we have obtained a new R that is incompressible, properly embedded, and such that ∂R is non-trivial in $H_1(\partial M)$, but with a non-trivial element in $H_1(T)$. Hence, there is a component R_i of R containing an essential curve in T, and such that ∂R_i is non-trivial in $H_1(\partial M)$. Then R_i is not a ∂-parallel annulus. This finishes Claim 1.

Before stating our second claim, let us introduce the following concept:

Definition 6.2. [11, Page 14] *Let S be a surface properly embedded in M, that is, $\partial S \subset \partial M$. A* ∂-compressing disk *$D \subset M$ is a disk such that ∂D consists of two arcs α and β such that $\alpha \cap \beta = \partial\alpha = \partial\beta$, where $\alpha = D \cap S$ and $\beta = D \cap \partial M$, see Figure 1.*

A properly embedded surface S is ∂-incompressible *if for each compressing disk D there is a disk $D' \subset S$ with $\alpha \subset \partial D'$ and $\partial D' \setminus \alpha \subset \partial S$, see Figure 2.*

Also recall the following property of incompressible surfaces that will be used in Claim 3;

Lemma 6.3. [11, Lemma 1.10] *Let N be a compact irreducible 3-manifold, such that ∂N consists of incompressible tori. If S is a connected incompressible surface*

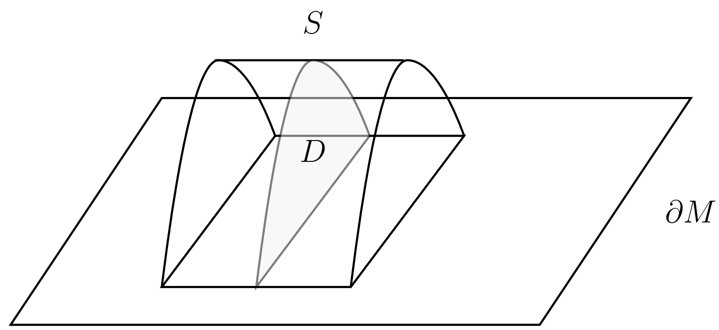

FIGURE 1. A ∂-compressing disk D

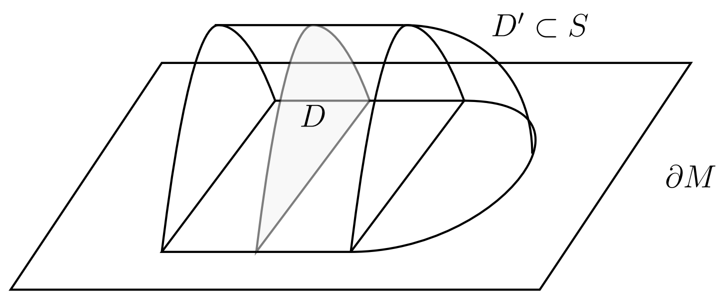

FIGURE 2. A ∂-incompressible surface S

properly embedded in N, then either S is a ∂-parallel annulus or else S is ∂-incompressible.

Consider now an Anosov torus $T \subset \partial M$, and let $f : M \to M$ be a diffeomorphism such that $f|T$ is a hyperbolic automorphism. Let S be the surface obtained in Claim 1, and let $\gamma \subset \partial S$ be an essential curve in T, as obtained in Claim 1. We loose no generality in assuming that γ is a line in T with rational slope.

Claim 2. *For some $n_0 > 0$ and $g : M \to M$ a suitable perturbation of f^{n_0}, there exists an annulus $A \subset S \cup g(S)$ such that one component of ∂A is a closed curve in T formed by one sub-arc γ_1 of γ and one sub-arc γ_2 of $g(\gamma)$. A is constructed in such a way that, after an isotopy, it is properly embedded.*

The number of disjoint non-isotopic properly embedded simple curves contained in S has an upper bound κ. By taking n sufficiently large, we can obtain $\#(\gamma \cap f^n(\gamma)) \geq 2(\kappa+1)^2$ (note that $\# f^n(\gamma) \cap \gamma$ goes to infinity as $n \to \infty$) We choose n_0 fixed satisfying the latter inequality.

Consider g a small perturbation of f^{n_0} such that $g|_T = f^{n_0}|_T$ and $g(S)$ is transverse to S. Observe that it is possible to obtain such a g since $f^{n_0}(S)$ is transverse to S when restricted to T. The way we choose n_0 implies that there are at least $(\kappa+1)^2$ curves in $S \cap g(S)$ with an extreme in γ. Then, $\kappa+1$ of them are isotopic in $(S, \partial S)$. Since the same κ is also an upper bound for the number of non-isotopic properly embedded simple curves contained in $g(S)$, at least two of these curves, say α_1 and α_2, are also isotopic in $(g(S), \partial g(S))$. Moreover, we choose α_1 and α_2 in such a way that if α is a curve of $S \cap g(S)$ between α_1 and α_2 then α is not isotopic to α_1 (neither α_2) in $(g(S), \partial g(S))$.

The annulus A is built in the following way. Construct a rectangle R by joining α_1 and α_2 by means of an arc $\gamma_1 \subset \gamma$ and and arc $\beta_1 \subset \partial S$. Construct another rectangle R' by joining α_1 and α_2 by means of an arc $\gamma_2 \subset g(\gamma)$ and an arc $\beta_2 \subset \partial g(S)$. This is possible since α_1 and α_2 are isotopic both in $(S, \partial S)$ and $(g(S), \partial g(S))$.

It remains to prove that, after an isotopy, A is properly embedded. The way we have chosen α_1 and α_2 implies that there are two possibilities for the curves in $R \cap R'$. Indeed, there are no curves in $R \cap R'$ joining the two boundaries of A then, either the curve is closed or its extreme points are both contained in the same boundary component of A. Irreducibility of M gives that closed curves can be eliminated by an isotopy. Suppose now that τ in $R \cap R'$ is an arc with both extremes in the same component of ∂A. Then, τ and an arc of ∂R bound a disk $D \subset R$. Analogously, we obtain $D' \subset R'$ bounded by τ and an arc of $\partial R'$. Thus, $\bar{D} = D \cup D'$ is a disk whose boundary is contained in ∂M and incompressibility of ∂M implies that $\partial \bar{D}$ bounds a disk in ∂M. Again, irreducibility of M allows us to isotope A in order to eliminate this type of intersections.

In this way we obtain a properly embedded annulus A proving Claim 2.

The rest of the section is devoted to proving that the annulus A obtained in Claim 2 is non-∂-parallel.

Claim 3. *The annulus A is non-∂-parallel.*

We begin by proving that A is incompressible. Indeed, we shall see that $\gamma_1 \cup \gamma_2 \subset A \cap T$ is an essential curve in T. Recall that γ is a line in T with rational slope. Since $f|T$ is a hyperbolic automorphism, $g(\gamma)$ is a line with different slope. The lifts of γ and $g(\gamma)$ to the universal covering cannot bound a region, as it is seen in Figure 3. Thus any closed curve formed by a segment in γ and a segment in $g(\gamma)$, like $\gamma_1 \cup \gamma_2$, is essential in T. This implies that A is incompressible.

So, it remains to prove that A is not ∂-parallel. Arguing by contradiction, suppose that A is ∂-parallel. This implies that $\partial A \subset T$, and also, that there exists a ∂-compressing disk D with $\alpha \subset \partial D$ for all arcs α properly embedded in A and with endpoints in different components of ∂A (see Figure 4). In particular, we can choose an arc $\alpha = \alpha_1$ as in the proof of Claim 2, that is, an arc $\alpha_1 \subset S \cap g(S)$, with one endpoint in $\gamma \cap g(\gamma)$ and the other endpoint in the other component of ∂A. The rest of the boundary of D, $\partial D \setminus \alpha_1$ is contained in T.

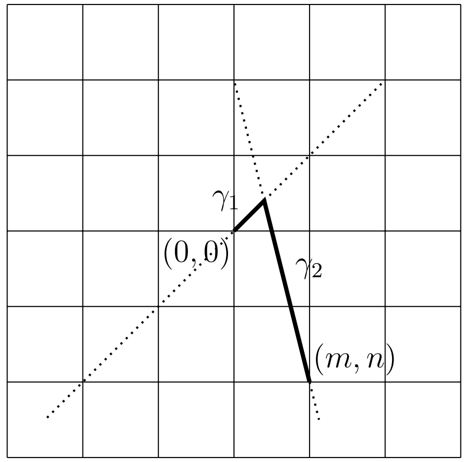

FIGURE 3. A closed curve in T formed by segments of different slope

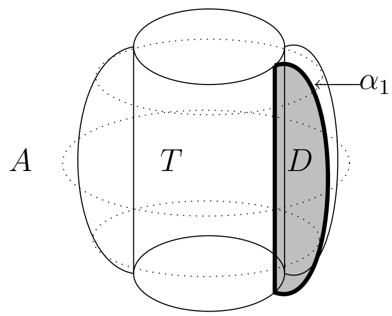

FIGURE 4. ∂-parallel annulus

Now, Claim 1 and Lemma 6.3 imply that both S and $g(S)$ are ∂-incompressible. Hence, since α_1 is properly embedded in S and in $g(S)$, we have that D is a ∂-compressing disk for S and for $g(S)$.

∂-incompressibility of S and $g(S)$ implies that there are two disks $D_1 \subset S$ and $D_2 \subset g(S)$ such that $\partial D_1 = \alpha_1 \cup l_1$, where l_1 is a segment in γ, and $\partial D_2 = \alpha_1 \cup l_2$, where l_2 is a segment in $g(\gamma)$.

The set $D' = D_1 \cup D_2$ is an immersed disk, and $\partial D' = l_1 \cup l_2$. Now, $l_1 \subset \gamma$ and $l_2 \subset g(\gamma)$. As we have said at the beginning of this proof, this implies that $\partial D'$ is essential in T, so D' cannot be a disk. This implies that A is non-∂-parallel and finishes Claim 3.

In conclusion, we have seen that the existence of an Anosov tori in the boundary of a 3-manifold whose boundary consists of incompressible tori, implies that the 3-manifold is not acylindrical. This proves Proposition 6.1.

7. Proof of Theorems 1.1 and 1.2

In this Section we finish Theorem 1.1, and prove Theorem 1.2. We begin by finishing the necessity part of Theorem 1.1.

Recall that Proposition 2.4 reduced the proof of the necessity part of Theorem 1.1 to three cases: case (3a): when the Anosov torus is part of the cutting tori of the JSJ-decomposition, case (3b): when the Anosov torus is in the interior of a Seifert component of the JSJ-decomposition, and is not ∂-parallel, and (3c): when M is a Seifert manifold and the Anosov torus is horizontal.

In Proposition 4.1, we show that in case (3c), then M is either the 3-torus, or else the mapping torus of $-id$. This proves Theorem 1.1 in this case.

In Proposition 5.1, we prove that if M is a compact connected irreducible Seifert manifold with or without boundary, admitting an Anosov torus T, then we have that either M is as in case (3c), namely, M is the 3-torus, or the mapping torus of $-id$; or else, M is $T \times [0,1]$. But in this last case, T is ∂-parallel. So, this proves Theorem 1.1 in case (3b).

The last case left is (3a), when the Anosov torus is one of the cutting tori of a minimal JSJ-decomposition. Proposition 6.1 shows that the Anosov torus cannot bound an atoroidal and acylindrical component of the JSJ-decomposition. Hence, the Anosov torus is a component of the boundary of a compact connected irreducible Seifert manifold. Proposition 5.1 shows that the Seifert component having the Anosov torus T as part of its boundary is T cross the interval. Hence the boundary of the Seifert component of the Anosov torus consists of two isotopic tori. Since the cutting tori of the JSJ-decomposition were taken to be a minimal family, this implies that the JSJ-decomposition consists of only one torus, the Anosov torus T. If we cut M along T we obtain T cross the interval. Let $A = f|_T$ be the hyperbolic automorphism obtained since T is an Anosov torus, and let $g : T \to T$ be a gluing diffeomorphism so that when we identify x with $g(x)$ we reobtain M. Then g commutes with A. This gives us three classes of diffeomorphisms: isotopic to $\pm id$, which would give us a Seifert M, or isotopic to a hyperbolic automorphism, which would give us that M is the mapping torus of a hyperbolic automorphism. In the first situations we would have that there are no cutting tori in the JSJ-decomposition, so we are in the last situation, and this finishes the proof of the necessity part of Theorem 1.1.

The sufficiency part, as we have said is straightforward. If M is the 3-torus, we can take $f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times id$ on $\mathbb{T}^2 \times \mathbb{S}^1$, this gives us infinitely many Anosov tori. If M is the mapping torus of a hyperbolic automorphism A over $\mathbb{T}^2$, there is a natural flow f_t which is the suspension of A. The time-one map of this flow f leaves invariant infinitely many tori, on which its dynamics is hyperbolic. Finally, let M be the mapping torus of $-id$. Cut M along an incompressible torus. We obtain a torus cross the interval. Define $g = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times id$ on this new manifold with boundary $\tilde{M} = \mathbb{T}^2 \times [0,1]$. To reobtain M we identify $\mathbb{T}^2 \times \{0\}$ with $\mathbb{T}^2 \times \{1\}$ by means of the map $(x, 0) \mapsto (-x, 1)$. But this map commutes with g on the boundary of $\tilde{M}$, hence g extends to a diffeomorphism on M leaving invariant infinitely many Anosov tori. This finishes the proof of Theorem 1.1.

Finally, let us prove Theorem 1.2. Let M be a compact orientable irreducible 3-manifold with non-empty boundary consisting of incompressible tori, and admitting an Anosov torus T. Duplicate M to obtain $M \sqcup M$ and glue along the corresponding boundary components. In this way we obtain a closed orientable irreducible 3-manifold N. Now, N is in the hypotheses of Theorem 1.1, so if we cut N along T we obtain T cross the interval. All the incompressible tori in the boundary of M are hence isotopic to T, this implies that M is in fact T cross the interval. This proves Theorem 1.2.

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IMERL-Facultad de Ingeniería, Universidad de la República, CC 30 Montevideo, Uruguay.
E-mail address: frhertz@fing.edu.uy
URL: http://www.fing.edu.uy/~frhertz

IMERL-Facultad de Ingeniería, Universidad de la República, CC 30 Montevideo, Uruguay.
E-mail address: jana@fing.edu.uy
URL: http://www.fing.edu.uy/~jana

IMERL-Facultad de Ingeniería, Universidad de la República, CC 30 Montevideo, Uruguay.
E-mail address: ures@fing.edu.uy
URL: http://www.fing.edu.uy/~ures