INTRINSIC ERGODICITY OF PARTIALLY HYPERBOLIC DIFFEOMORPHISMS WITH HYPERBOLIC LINEAR PART

RAÚL URES

ABSTRACT. We prove that any (absolutely) partially hyperbolic diffeomorphism f of $\mathbb{T}^3$ homotopic to a hyperbolic automorphism A is intrinsically ergodic, that is, it has a unique entropy maximizing measure μ.

1. INTRODUCTION

A diffeomorphism $f : M \to M$ of a closed smooth manifold M is partially hyperbolic if TM splits into three invariant bundles such that one of them is contracting, the other is expanding, and the third, called the center bundle, has an intermediate behavior, that is, not as contracting as the first, nor as expanding as the second. In this paper we consider (absolutely) partially hyperbolic diffeomorphisms with one-dimensional center bundle homotopic to a hyperbolic automorphism of $\mathbb{T}^n$ (see Section 2 for precise definitions). Our main issue will be the study of the properties of the entropy maximizing measures for these systems.

Entropies are quantities that measure the complexity of the orbits of a system. While the topological entropy "sees" the whole complexity of the orbits of a system, the metric entropy "sees" the complexity of the orbits that are relevant for a given measure. An entropy maximizing measure (or just a maximizing measure) is an ergodic measure such that its metric entropy equals the topological entropy of the system. In the previous informal words, the complexity of the orbits that are seen by a maximizing measure is the same as the complexity of the orbits of the whole system.

On the one hand, it is well-known that uniformly hyperbolic systems admit maximizing measures and their topological transitivity implies uniqueness. See the works of R. Bowen [1] and G. Margulis [14]. On the other hand, when the system is not hyperbolic the existence can fail if it is not smooth enough (see [15] and [16]). However, if we are in the setting of partially hyperbolic diffeomorphism, the results of W. Cowieson and L.-S. Young in [6] (see also [7]) imply that there are always entropy maximizing measures if the center bundle is one-dimensional even if the diffeomorphism is C^1.

Although existence is already provided by Cowieson-Young's results our method gives it immediately as a consequence of the properties of the semiconjugacy of f with its linear part. Our main result is the uniqueness of the entropy maximizing measure. B. Weiss [21] called the systems having this property intrinsically ergodic, that is, f is intrinsically ergodic if it has a unique maximizing measure. Let us state our main result.

THEOREM 1.1. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be an absolutely partially hyperbolic diffeomorphism homotopic to a hyperbolic linear automorphism. Then, f is intrinsically ergodic.*

The proof of this theorem depends on some new results about partially hyperbolic diffeomorphisms. The first one of these results is that an absolutely partially hyperbolic diffeomorphism of $\mathbb{T}^3$ has quasi-isometric strong foliations (see [3]). The second result is a byproduct of A. Hammerlindl's leaf conjugacy (see [9]). Hammerlindl showed that the quasi-isometry property for strong foliations implies the quasi-isometry property for the center foliation (see Lemma 3.3).

After Brin-Burago-Ivanov's result [3] (see Theorem 2.1), Theorem 1.1 is a direct consequence of the following.

THEOREM 1.2. *Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be an absolutely partially hyperbolic diffeomorphism with one-dimensional center having quasi-isometric strong foliations homotopic to a hyperbolic linear automorphism A. Then, f is intrinsically ergodic.*

Moreover, if μ is the maximizing measure of f and m is the Lebesgue measure, (f, μ) and (A, m) are isomorphic.

This work is complementary to the joint paper of the author with F. Rodriguez Hertz, M. A. Rodriguez Hertz and A. Tahzibi [18] where the maximizing measures of partially hyperbolic diffeomorphisms of 3-dimensional manifolds having compact center leaves are studied. In contrast to Theorem 1.1, if the center leaves are compact, uniqueness of maximizing measures holds for a meager set of diffeomorphisms (see comments below).

Let us mention the few related results known for partially hyperbolic diffeomorphisms. In [17] S. Newhouse and L. S. Young have shown that some partially hyperbolic examples of $\mathbb{T}^4$ have a unique entropy maximizing measure (see for instance the examples in [20]). In 3-dimensional manifolds, J. Buzzi, T. Fisher, M. Sambarino and C. Vásquez [5] proved that certain explicit DA construction due to R. Mañé (see [13]) also has this uniqueness property. This result can also be obtained as a consequence of Theorem 1.2. Finally, there is [18] where the authors studied, as we mentioned before, the case where the center leaves are compact, the manifold is 3-dimensional and f is accessible. They obtained the following dichotomy: either the center Lyapunov exponent is zero and there is a unique entropy maximizing measure or there are at least two (but finite) maximizing measures. The first case holds for a meager set of diffeomorphisms because if f is in this set, it is topologically conjugate to a rotation extension of an Anosov diffeomorphism. In particular, if we are in the first case, there are no hyperbolic periodic points.

Our results go beyond the existence and uniqueness of the maximizing measure μ and give a detailed description of some properties of μ. The aim of this description is to show that the system is, in some sense, "intrinsically Anosov". On the one hand, in Section 5 we shall prove, based on a Pesin-Ruelle-like inequality obtained by Y. Hua, R. Saghin and Z. Xia [10], that the center exponent has absolute value greater or equal than the center exponent of its linear part if we are in the conditions of Theorem 1.1. Observe that, in particular, this implies the presence of many hyperbolic periodic points with large center eigenvalue. On the other hand, in Section 6 we shall show that the strong stable foliation has a unique minimal set if the center Lyapunov exponent of the linear part is positive. Moreover, this minimal set is the support of μ. We conjecture that this set is the whole manifold which would imply that the diffeomorphisms satisfying the hypothesis of Theorem 1.2 are topologically transitive (even mixing).

The paper is organized as follows. In Section 2 we present definitions and some previous results needed for the rest of the paper. Sections 3 and 4 are devoted to the proof of Theorem 1.2. In Section 3 we prove some properties of the semiconjugacy h between f and its linear part. In particular, we show that h can only collapse center arcs. In Section 4 we show how the properties of h and Ledrappier-Walters' formula [12] imply uniqueness of the maximizing measure obtaining the proof of Theorem 1.2 at the end of the section. As we have already

mentioned, Section 5 is devoted to show that the center exponent is nonzero and in Section 6 we prove that $\text{supp}(\mu)$ is the unique minimal set of the strong stable foliation (provided that the center exponent is positive). Finally, in Section 7 we make some comments on the A. Katok's conjecture about the existence of measures with intermediate entropies and show that the conjecture is true in the setting of this paper.

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2. Preliminaries

Throughout this paper we shall work with a ***partially hyperbolic diffeomorphism*** f, that is, a diffeomorphism admitting a nontrivial Tf-invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$, such that all unit vectors $v^\sigma \in E_x^\sigma$ ($\sigma = s, c, u$) with $x \in M$ satisfy:

$$\|T_x f v^s\| < \|T_x f v^c\| < \|T_x f v^u\|$$

for some suitable Riemannian metric. f also must satisfy that $\|Tf|_{E^s}\| < 1$ and $\|Tf^{-1}|_{E^u}\| < 1$. In fact we will use a stronger type of partial hyperbolicity. We will say that f is ***absolutely partially hyperbolic*** if it is partially hyperbolic and

$$\|T_x f v^s\| < \|T_y f v^c\| < \|T_z f v^u\|$$

for all $x, y, z \in M$ and $v^\sigma \in E_w^\sigma$ unit vectors, $\sigma = s, c, u$ and $w = x, y, z$ respectively.

For partially hyperbolic diffeomorphisms, it is a well-known fact that there are foliations $\mathcal{W}^\sigma$ tangent to the distributions E^σ for $\sigma = s, u$. The leaf of $\mathcal{W}^\sigma$ containing x will be called $W^\sigma(x)$, for $\sigma = s, u$.

In general it is not true that there is a foliation tangent to E^c. It can fail to be true even if $\dim E^c = 1$ (see [19]). We shall say that f is ***dynamically coherent*** if there exist invariant foliations $\mathcal{W}^{c\sigma}$ tangent to $E^{c\sigma} = E^c \oplus E^\sigma$ for $\sigma = s, u$. Note that by taking the intersection of these foliations we obtain an invariant foliation $\mathcal{W}^c$ tangent to E^c that subfoliates $\mathcal{W}^{c\sigma}$ for $\sigma = s, u$. In $\mathbb{T}^3$, Brin, Burago and Ivanov have shown the following:

THEOREM 2.1 ([3]). *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be an absolutely partially hyperbolic diffeomorphism then it is dynamically coherent.*

This result is a consequence of a more general approach to the subject. A foliation $\mathcal{W}$ of a simply connected Riemannian manifold is quasi-isometric if there are $a, b \in \mathbb{R}$ such that $d_W(x, y) \le a\, d(x, y) + b$ for any x, y in the same leaf W of $\mathcal{W}$. Here d_W stands for the distance induced by the restriction to W of the ambient Riemannian metric. Brin, Burago and Ivanov showed that in $\mathbb{T}^3$ the strong foliations are quasi-isometric in the universal cover. Then, Theorem 2.1 is a consequence of the following result of Brin:

THEOREM 2.2 ([2]). *Let f be a partially hyperbolic diffeomorphism of a compact n-dimensional Riemannian manifold M. Suppose the stable and unstable foliations of f are quasi-isometric in the universal cover $\tilde{M}$. Then, f is dynamically coherent.*

Let $n \in \mathbb{N}$ and $\delta > 0$. A finite subset E is (n, δ)-separated if for $x, y \in E$, $x \ne y$, we have $\max_{i=0,\dots,n} d(f^n(x), f^n(x)) \ge \delta$. Let

$$h_n(f, K, \delta) = \sup\{\#E; E \subset K \text{ is } (n, \delta) - \text{separated}\}$$

and
$$h(f, K) = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log h_n(f, K, \delta).$$

When $K = M$ call $h(f, M) = h_{top}(f)$ the **topological entropy** of f. It is a well-known fact that $h(f, K) = 0$ if K is a subset of a curve having all its iterates with uniformly bounded length.

Throughout the paper we shall suppose that the definition of entropy of an $f-$invariant probability μ, $h_\mu(f)$, is known among other basic concepts of ergodic theory.

The variational principle states that $\sup\{h_\mu(f); \mu \text{ is } f\text{-invariant}\} = h_{top}(f)$ if f is a continuous map of a compact metric space. We will say that an invariant probability measure μ satisfying $h_\mu(f) = h_{top}(f)$ is an **entropy maximizing measure** or shortly a **maximizing measure**.

3. Properties of the semiconjugacy

Let f be an absolutely partially hyperbolic diffeomorphism homotopic to a hyperbolic automorphism A of $\mathbb{T}^n$, with one-dimensional center and with quasi-isometric strong foliations. By a well-known result of Franks [8] f is semiconjugate to A. More specifically, there exists $h : \mathbb{T}^n \to \mathbb{T}^n$ homotopic to the identity such that $A \circ h = h \circ f$. This equality can be expressed in $\mathbb{R}^n$, the universal cover of $\mathbb{T}^n$, by taking lifts in an adequate way obtaining $\tilde{A} \circ \tilde{h} = \tilde{h} \circ \tilde{f}$ with $\tilde{h}$ at bounded distance of the identity map. In this section we shall focus on the study of the sets where the injectivity of $\tilde{h}$ fails. In fact, we shall prove the following result:

PROPOSITION 3.1. *For all $z \in \mathbb{R}^n$, $\tilde{h}^{-1}(z)$ is a compact connected subset (i.e. an arc or a point) of a center manifold.*

We shall postpone a little bit the proof of this proposition.

One of the most important properties of $\tilde{h}$ is that $\tilde{h}(\tilde{x}) = \tilde{h}(\tilde{y})$ if and only if there exists $K > 0$ such that $d(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{y})) < K$ for all $n \in \mathbb{Z}$. Moreover, K can be taken to be independent of $\tilde{x}$ and $\tilde{y}$.

Before going into the proof of Proposition 3.1 we shall prove the following lemma.

LEMMA 3.2. *Suppose that $\tilde{h}(\tilde{x}) = \tilde{h}(\tilde{y})$. Then, $\tilde{y} \in W^c(\tilde{x})$.*

Proof. If $\tilde{y} \notin W^c(\tilde{x})$ we have that either $\tilde{y} \notin W^{cs}(\tilde{x})$ or $\tilde{y} \notin W^{cu}(\tilde{x})$. Suppose that we are in the first case (the other one being analogous). Let $\tilde{z} = W^u(\tilde{y}) \cap W^{cs}(\tilde{x})$ and call $D_{cs} = d_{cs}(\tilde{x}, \tilde{z})$ and $D_u = d_u(\tilde{y}, \tilde{z})$. The existence (and uniqueness) of $\tilde{z}$ was proved in [3] for $\mathbb{T}^3$ and Proposition 2.15 of Hammerlindl thesis [9] in our more general setting. Now, the absolute partial hyperbolicity implies the existence of constants $1 < \lambda_c < \lambda_u$ such that $\forall n > 0$ $d(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{z})) \leq \lambda_c^n D_{cs}$ and $d_u(\tilde{f}^n(\tilde{y}), \tilde{f}^n(\tilde{z})) \geq \lambda_u^n D_u$. Since $\mathcal{W}^u$ is quasi isometric we have that $d(\tilde{f}^n(\tilde{y}), \tilde{f}^n(\tilde{z})) > \frac{1}{a}(\lambda_u^n D_u - b)$. Finally, $d(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{y})) > \frac{1}{a}(\lambda_u^n D_u - b) - \lambda_c^n D_{cs}$. This quantity goes to infinity with n implying that $\tilde{h}(\tilde{x}) \neq \tilde{h}(\tilde{y})$ and finishing the proof of the lemma. $\square$

The following lemma proved in [9] will also be useful.

LEMMA 3.3 (Hammerlindl, [9]). *$W^c(f)$ is quasi-isometric in the universal cover ($\mathbb{R}^n$).*

Now we are ready to prove the proposition.

Proof of Proposition 3.1. $\tilde{h}^{-1}(z)$ is a compact set contained in a center manifold by Lemma 3.2. Take $\tilde{x}, \tilde{y} \in \tilde{h}^{-1}(z)$. On the one hand, we know that there exists K such that $d(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{y})) < K$. On the other hand, if we take $\tilde{w}$ in the center segment joining $\tilde{x}$ and $\tilde{y}$, the quasi-isometry property of the center foliation implies that there are constants $a, b > 0$ such that, $\forall n \in \mathbb{Z}$,

$$d(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{w})) \le d_{W^c}(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{w})) \le d_{W^c}(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{y})) \le ad(\tilde{f}^n(\tilde{x}), \tilde{f}^n(\tilde{y})) + b \le aK + b.$$

Then, $\tilde{h}(\tilde{w}) = \tilde{h}(\tilde{x}) = \tilde{h}(\tilde{y})$ implying that the whole center arc joining $\tilde{x}$ and $\tilde{y}$ is contained in $\tilde{h}^{-1}(z)$. Thus, $\tilde{h}^{-1}(z)$ is connected. $\qquad\square$

At this point we need to show that A has also a partially hyperbolic splitting similar to the partially hyperbolic splitting of f. This was already proved in [3] and [9] but in our setting the proof is easier and, for the sake of completeness, we include an outline of the proof.

Let $\rho, \rho' > 1$ and $\delta, \delta' < 1$ be such that

$$\|T_x f v^s\| < \delta < \delta' < \|T_y f v^c\| < \rho' < \rho < \|T_z f v^u\|$$

for all $x, y, z \in M$ and $v^\sigma \in E_w^\sigma$ unit vectors, $\sigma = s, c, u$ and $w = x, y, z$ respectively.

LEMMA 3.4. $\tilde{A}$ has a partially hyperbolic splitting $E_A^s \oplus E_A^c \oplus E_A^u = \mathbb{R}^n$ such that $\dim(E_A^c)$ is one dimensional and $\delta' < \tilde{A}v^c < \rho'$ if v^c is a unit vector in E_A^c.

Proof. Let $\tilde{x}$ be a fixed point for $\tilde{f}$. $\tilde{h}(W^c(\tilde{x}))$ is an $\tilde{A}-$invariant curve and therefore is contained in the stable or unstable manifold of 0 (the unique fixed point of $\tilde{A}$). Suppose that $\tilde{h}(W^c(\tilde{x})) \subset W_{\tilde{A}}^u(0)$, the other case being analogous. On the one hand, the distance to 0 of any point $\tilde{z}$ in $\tilde{h}(W^c(\tilde{x}))$ grows less that $C(\rho')^n + K$ under $\tilde{A}-$iteration; where C, K are some positive constants. Indeed the length of a center curve joining $\tilde{x}$ and an $\tilde{h}-$preimage of $\tilde{z}$ grows less than $C(\rho')^n$ and $\tilde{h}$ is within a finite distance of the identity. This implies that $\tilde{A}$ has, at least, one unstable eigenvalue with modulus less that ρ'. On the other hand, $\tilde{h}(W^u(\tilde{x})) \subset W_{\tilde{A}}^u(0)$ and a similar argument gives that distances grow more than $C\rho^n + K$. Then, it is not difficult to conclude that there is a unique unstable eigenvalue with modulus less than ρ' and this gives the desired splitting.

$\qquad\square$

REMARK 3.5. *Let us make some observations.*

(1) *The arguments of Lemma 3.4 imply that the $h-$image of a center manifold of f is a center manifold (line) of A.*

(2) *Conversely, observe that Proposition 3.1 (actually Lemma 3.2) implies that $\tilde{h}^{-1}(W_{\tilde{A}}^c(\tilde{x}))$ is contained in $W_{\tilde{f}}^c(\tilde{z})$ for any $\tilde{z} \in \tilde{h}^{-1}(\tilde{x})$. Moreover, we have that $\tilde{h}^{-1}(W_{\tilde{A}}^c(\tilde{x})) = W_{\tilde{f}}^c(\tilde{z})$.*

4. UNIQUENESS

In this section f will continue to be an absolutely partially hyperbolic diffeomorphism homotopic to a hyperbolic automorphism A of $\mathbb{T}^n$, with one-dimensional center and with quasi-isometric strong foliations. We shall prove the uniqueness of the entropy maximizing measure of f obtaining Theorem 1.2.

Let $\tilde{X} = \{\tilde{x} \in \mathbb{R}^n; \#\tilde{h}^{-1}(\tilde{x}) > 1\}$. By the results of the previous section $\tilde{X}$ is the set of points whose $h-$preimages are nontrivial center arcs. Let $\pi : \mathbb{R}^n \to \mathbb{T}^n$ be the covering projection and $X = \pi(\tilde{X})$.

LEMMA 4.1. $m(\tilde{X}) = 0$ *(of course we also have that $m(X) = 0$)*

Proof. Let $W_{\tilde{A}}^c(\tilde{x})$ be a center manifold for A that is one dimensional by Lemma 3.4. Remark 3.5 says that there exists $\tilde{z}$ such that $\tilde{h}^{-1}(W_{\tilde{A}}^c(\tilde{x})) = W_{\tilde{f}}^c(\tilde{z})$. Let $\tilde{X}_{\tilde{x}}^c = W_{\tilde{A}}^c(\tilde{x}) \cap \tilde{X}$. Since $\tilde{h}^{-1}(\tilde{y})$ is a nontrivial interval of $W_{\tilde{f}}^c(\tilde{z})$ for all $\tilde{y} \in \tilde{X}_{\tilde{x}}^c$, we have that $\tilde{X}_{\tilde{x}}^c$ is a countable set. We

obtained that $\tilde{X}_{\tilde{x}}^c$ is countable $\forall \tilde{x} \in \mathbb{R}^n$ and since $\tilde{A}$ is linear we know that the center foliation of A is a foliation by parallel straight lines. So, Fubini's Theorem gives $m(\tilde{X}) = 0$. $\qquad\square$

Certainly, the previous lemma implies the existence and uniqueness of a measure μ whose $h-$image is m (existence was already known as we have already said in the Introduction). Thanks to the Rokhlin's disintegration it is also true that any $f-$invariant measure goes via h onto an $A-$invariant measure. Thus, to obtain a proof of Theorem 1.2 it is enough to prove that the $h-$image of an entropy maximizing measure is the Lebesgue measure m. Essentially this is a consequence of the following Ledrappier-Walters formula and our previous results. Take any invariant measure ν for f and let $\hat{\nu} = \nu \circ h^{-1}$ (observe that $\hat{\nu}$ is given by Rokhlin disintegration). Then, Ledrappier-Walters variational principle [12] says

$$\sup_{\tilde{\mu}:\tilde{\mu}\circ h^{-1}=\hat{\nu}} h_{\tilde{\mu}}(f) = h_{\hat{\nu}}(A) + \int_{\mathbb{T}^n} h(f, h^{-1}(y))d\hat{\nu}(y).$$

Proof of Theorem 1.2. First observe that, since the $h-$preimages of points are arcs of uniformly bounded length and the partition by $h-$preimages is $f-$invariant, we obtain that $h(f, h^{-1}(y)) = 0$ for any $y \in \mathbb{T}^n$. This means, after Ledrappier-Walters formula, that $h_\nu(f) = h_{\hat{\nu}}(A)$ for any $f-$invariant measure ν. As an easy consequence we have that any entropy maximizing measure for f has as its $h-$image the Lebesgue measure m, i.e. the unique entropy maximizing measure for A. The previous comments imply the uniqueness of such a measure giving the proof of Theorem 1.2. That (f, μ) and (A, m) are isomorphic via h is clear from the construction. $\qquad\square$

5. The center exponent

Let μ be the unique maximizing measure for an absolutely partially hyperbolic diffeomorphism homotopic to a hyperbolic automorphism A of $\mathbb{T}^n$, with one-dimensional center and with quasi-isometric strong foliations. We have shown that (f, μ) is isomorphic to (A, m). This shows that looking to the maximizing measure we recover many important dynamical properties of Lebesgue measure as an invariant measure for A. For instance we have that μ has the Bernoulli property. In this section we shall study the center Lyapunov exponent of μ. In order to recover the whole Anosov-like behavior we need to show that the center Lyapunov exponent of μ is nonzero. This property is not the main issue of this paper and we have found some difficulties in trying to state a relationship between the quasi-isometry of the strong leaves and its volume when the dimension is greater than one. Then, we shall prove our result for diffeomorphisms on $\mathbb{T}^3$.

THEOREM 5.1. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a $C^{1+\alpha}$ absolutely partially hyperbolic diffeomorphism homotopic to a hyperbolic linear automorphism A with center Lyapunov exponent $\lambda^c(A) > 0$. Let μ be the maximizing measure of f. Then, the center Lyapunov exponent of μ, $\lambda^c(\mu)$, satisfies*

$$\lambda^c(\mu) \geq \lambda^c(A).$$

Of course we have an analogous result if $\lambda^c(A) < 0$.

The proof of this result is based in a Pesin-Ruelle-like inequality proved by Y. Hua, R. Saghin and Z. Xia in [10]. Before going into the proof, let us introduce their result. Let $\mathcal{W}$ be a foliation (in this paper it will be the strong stable or the strong unstable foliation). Let $W_r(x)$ be the ball of the leaf $W(x)$ with radius r and centered at x. Let

$$\chi_{\mathcal{W}}(x, f) = \limsup_{n\to\infty} \frac{1}{n} \log(\text{Vol}(f^n(W_r(x)))),$$

$\chi_{\mathcal{W}}(x, f)$ is the volume growth rate of the foliation at x. Let

$$\chi_{\mathcal{W}}(f) = \sup_{x \in M} \chi(x, f).$$

Then, $\chi_{\mathcal{W}}(f)$ is the maximum volume growth rate of $\mathcal{W}$ under f. Let us denote $\chi_u(f) = \chi_{\mathcal{W}^u}(f)$ when f is a partially hyperbolic diffeomorphism. Then, the Hua-Saghin-Xia's result is the following.

THEOREM 5.2 ([10]). *Let f be a $C^{1+\alpha}$ partially hyperbolic diffeomorphism. Let ν be an ergodic measure and $\lambda_i^c(\nu)$ the Lyapunov exponents corresponding to E^c. Then,*

$$h_\nu(f) \leq \chi_u(f) + \sum_{\lambda_i^c > 0} \lambda_i^c(\nu).$$

With these ingredients we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. The first step is to estimate $\chi_u(f)$. Since $\mathcal{W}^u$ is 1-dimensional the volume is the length. Then, consider

$$\frac{1}{n} \log(\text{length}(f^n(W_r^u(x)))).$$

First of all observe that $\chi_u(f) = \chi_u(\tilde{f})$ where $\tilde{f}$ is any lift of f to universal cover. On the one hand, since $\mathcal{W}^u$ is quasi-isometric, we have that

$$\frac{1}{n} \log(\text{length}(\tilde{f}^n(W_r^u(\tilde{x})))) \leq \frac{1}{n} \log(a \operatorname{diam}(\tilde{f}^n(W_r^u(\tilde{x}))) + b)$$

for some constants $a, b > 0$.

On the other hand, $\tilde{h}(\tilde{f}^n(W_r^u(\tilde{x}))) = \tilde{A}^n(\tilde{h}(W_r^u(\tilde{x})))$. Let $C = \operatorname{diam}(\tilde{h}(W_r^u(\tilde{x})))$. Then, $\operatorname{diam}(\tilde{A}^n(\tilde{h}(W_r^u(\tilde{x})))) \leq C \exp(n\lambda^u(A))$ where $\lambda^u(A)$ is the strong unstable Lyapunov exponent of A. Since $\tilde{h}$ is at bounded distance from the identity we have that there exists a constant K such that $\operatorname{diam} \tilde{f}^n(W_r^u(\tilde{x})) \leq C \exp(n\lambda^u(A)) + K$. Thus,

$$\frac{1}{n} \log(\text{length}(\tilde{f}^n(W_r^u(\tilde{x})))) \leq \frac{1}{n} \log(a \operatorname{diam}(\tilde{f}^n(W_r^u(\tilde{x}))) + b) \leq$$

$$\leq \frac{1}{n} \log(a(C \exp(n\lambda^u(A)) + K) + b) \to \lambda^u(A).$$

Then, $\chi_u(f) \leq \lambda^u(A)$.

Secondly, we have that

$$\lambda^c(A) + \lambda^u(A) = h_{top}(A) = h_{top}(f) = h_\mu(f) \leq \chi_u(f) + \lambda^c(\mu).$$

So

$$\lambda^c(A) + \lambda^u(A) \leq \chi_u(f) + \lambda^c(\mu) \leq \lambda^u(A) + \lambda^c(\mu)$$

and then,

$$\lambda^c(A) \leq \lambda^c(\mu)$$

proving the theorem.

$\square$

After this result and the comments at the beginning of this section, a natural question is the following.

QUESTION 5.3. *Does Theorem 5.1 remain true in higher dimensions?*

6. Geometry of the support

In this section f will be again an absolutely partially hyperbolic diffeomorphism homotopic to a hyperbolic automorphism A of $\mathbb{T}^n$, with one-dimensional center and with quasi-isometric strong foliations. By Theorem 1.2 we know that f has a unique maximizing measure μ. We want to describe the support of μ. In this section assume that the center eigenvalue (see Lemma 3.4) of A has modulus greater that one (if not, take f^{-1}). We shall show that $\operatorname{supp}(\mu)$ is saturated by strong stable manifolds ($s-$saturated). That is, $W^s(x) \subset \operatorname{supp}(\mu)$ if $x \in \operatorname{supp}(\mu)$. Moreover, we will show that the strong stable foliation $\mathcal{W}^s$ has a unique minimal set and this minimal set is $\operatorname{supp}(\mu)$. Recall that a compact $s-$saturated set Γ is minimal if given $x \in \Gamma$ we have that $W^s(x)$ is dense in Γ.

Before showing the announced properties of $\operatorname{supp}(\mu)$ we need to prove some results.

PROPOSITION 6.1. $\operatorname{supp}(\mu)$ *is* $s-$*saturated.*

Proof. The first step is that $h(x) = h(y)$ implies $h(W^s(x)) = h(W^s(y)) = W_A^s(h(x))$. Observe that this property is not true for the strong unstable manifolds. Indeed the $h-$image of a strong unstable manifold of f in general is not a strong unstable manifold of A (recall that the situation is not symmetric, the center eigenvalue of A has modulus greater than one) [1] .

Our second step is to take x in the complement of $\operatorname{supp}(\mu)$. This means that there is a neighborhood U of x such that $\mu(U) = 0$. This is equivalent to the following fact: if $x \in U$ and σ is an arc in $U \cap W^c(x)$ then $h(\sigma)$ is a singleton. Indeed suppose $h^{-1}(h(x)) = x$. Suppose that V is a foliation chart for $\mathcal{W}^c$ such that $x \in V$. Then, if V is small enough, continuity of h implies that any maximal connected center arc $\mathcal{C}$ contained in V satisfies that there is $y \in \mathcal{C}$ such that $h^{-1}(h(y)) = y$. In particular, $h(\mathcal{C})$ is not singleton. Now, Fubini's easily gives that $m(h(V)) \neq 0$ and so, $\mu(V) \neq 0$. The converse, that is the fact that $\mu(U) \neq 0$ implies the existence of a point x such that $h^{-1}(h(x)) = x$, is easier.

Now take $W^s(U)$ the $s-$saturation of U. Recall that $h(x) = h(y)$ implies that the whole $h-$image of the center curve joining x and y is a unique point. Then, the first step of the proof gives us that any point $z \in W^s(U)$ has a neighborhood where all center curves are collapsed by h. This means that z is in the complement of the support of μ. Then, the complement of $\operatorname{supp}(\mu)$ is $s-$saturated finishing the proof of the proposition.   □

The following is also an important property.

LEMMA 6.2. *Let* Λ *be a closed nonempty* $s-$*saturated and* $f-$*invariant set. Then,* $\operatorname{supp}(\mu) \subset \Lambda$.

Proof. Since the $h-$image of a strong stable manifold is a stable manifold of A we have that $h(\Lambda) = \mathbb{T}^n$. It is not difficult to conclude that $f|_\Lambda$ has an invariant measure that goes via h to Lebesgue. Uniqueness of the entropy maximizing measure implies that this measure is μ. This clearly proves the lemma.   □

REMARK 6.3. *Observe that the previous lemma remains valid if* Λ *is* f^n-*invariant for some* n. *Indeed* f^n *has unique maximizing measure, and this measure is* μ. *Thus, for instance, the closure of the strong stable manifold of a periodic point contains* $\operatorname{supp}(\mu)$.

The next observation is the minimality of the center-unstable foliation. This is a consequence of Hammerlindl's leaf conjugacy but in our case the proof is more direct.

LEMMA 6.4. $\mathcal{W}^{cu}$ *is a minimal foliation.*

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[1]Even in the simpler case where f is conjugated to A and M is three-dimensional, we will obtain that the strong stable and strong unstable foliations would be jointly integrable if the h-images of strong unstable manifolds of f be strong unstable manifolds of A. This is not generic (see [4]).

Proof. It is an easy consequence of the following three facts:

(1) $h^{-1}(W_A^{cu}(x)) = W^{cu}(h^{-1}(x))$: On the one hand, since h only identifies points that belong to the same center, we have that if $W^{cu}(y) \neq W^{cu}(z)$ then $h(W^{cu}(y)) \neq h(W^{cu}(z))$. On the other hand, $h(W^{cu}(h^{-1}(x))) = W_A^{cu}(x)$. These two observations imply the first fact.

(2) $W_A^{cu}(x)$ is dense: $W_A^{cu}(x)$ is the unstable manifold of a linear hyperbolic automorphisms of $\mathbb{T}^n$.

(3) $h|_{W^s(z)}$ is injective for all z: This is a corollary of Lemma 3.2 and the fact that no strong stable manifold intersects twice the same center manifold.

Then, since a leaf of $\mathcal{W}_A^{cu}$ intersects densely each stable segment, the injectivity of $h|_{W^s(z)}$ gives that any center unstable leaf intersects any strong stable segment densely. $\square$
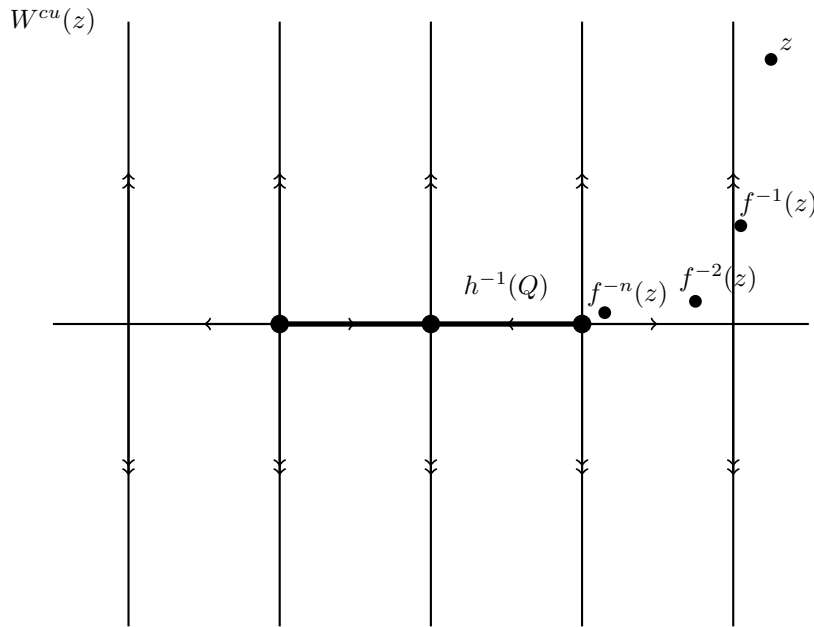
The same argument proves the corresponding lemma for $\mathcal{W}^{cs}$.

FIGURE 1. The α-limit of z

THEOREM 6.5. *$\mathcal{W}^s$ has a unique minimal set. This set is $\mathrm{supp}(\mu)$.*

Proof. Let us first show there are periodic points in $\mathrm{supp}(\mu)$. Take $W^{cu}(y)$ to be a periodic center unstable leaf. It is easy to see that there exist periodic leaves because $h-$preimages of $A-$periodic leaves are periodic.

Now, since $W^{cu}(y)$ is dense and $\mathrm{supp}\,\mu$ is $s-$saturated there is a point $z \in \mathrm{supp}\,\mu \cap W^{cu}(y)$. The $\alpha-$limit of z ($\alpha(z)$) is in $\mathrm{supp}(\mu)$ by compactness and invariance. We shall show $\alpha(z)$ is a periodic point. Since h is a semiconjugacy we have that $h(\alpha(z)) \subset \alpha(h(z))$. $W^{cu}(y)$ is periodic so, $h(W^{cu}(y)) = h(W^{cu}(z)) = W_A^{cu}(h(z))$ is periodic too. $W_A^{cu}(h(z))$ is an unstable manifold of A and thus, $\alpha(h(z))$ is a periodic point. Suppose, for simplicity, that $\alpha(h(z)) = Q$ is a fixed point. So, $\alpha(z) \subset h^{-1}(Q)$ and $h^{-1}(Q)$ is an invariant center arc contained in $W^{cu}(z)$. This implies that $\alpha(z)$ is a periodic point (see Figure 1).
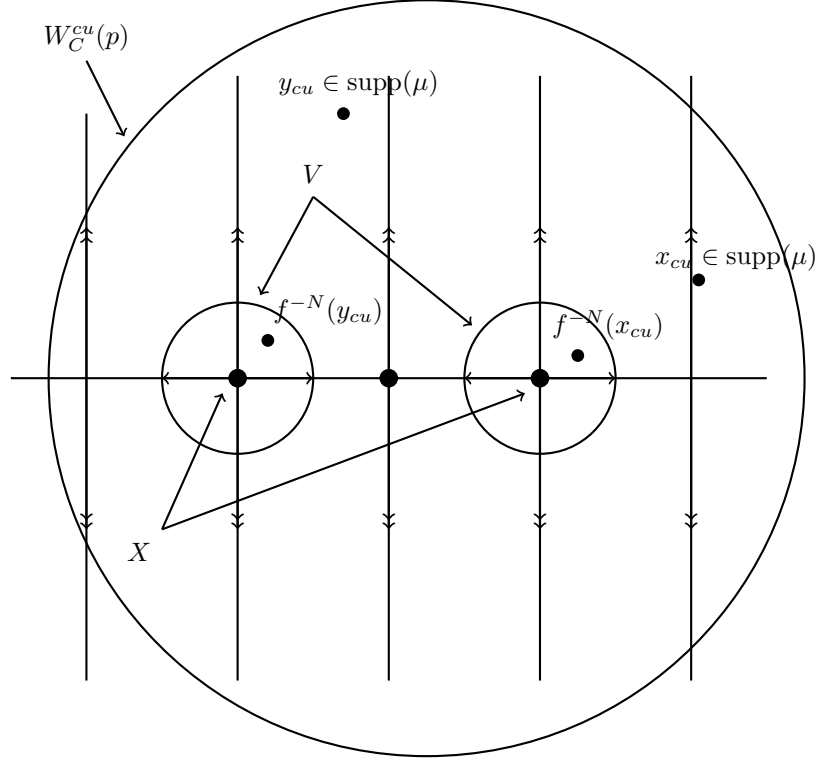
FIGURE 2.   The points of $\mathrm{supp}(\mu) \cap W_C^{cu}(p)$ enter simultaneously V.

Then, $\mathrm{supp}(\mu)$ has a lot of periodic points.

Suppose that $p \in \mathrm{supp}(\mu)$ is a fixed point. In our case, there is always a fixed point but, anyway, if p is just periodic, the argument essentially does not change thanks to Remark 6.3. On the one hand, the fact that $\mathrm{supp}(\mu)$ is s-saturated (Lemma 6.1) and Lemma 6.2 imply $\overline{W^s(p)} = \mathrm{supp}(\mu)$. On the other hand, there exists $C > 0$ such that $W^s(x) \cap W_C^{cu}(p) \neq \emptyset$ $\forall x \in \mathbb{T}^n$. Here $W_C^{cu}(p)$ is the set of points belonging to $W^{cu}(p)$ that are a distance less than C with the distance induced by the restriction of the ambient Riemannian metric.

Let $z \in \mathrm{supp}(\mu) \cap W^{cu}(p)$. Recall that $\alpha(z)$ is a periodic point contained in $\mathrm{supp}(\mu) \cap W^c(p)$. Call $X = \mathrm{supp}(\mu) \cap W^c(p) \cap \mathcal{P}er(f)$. X is a compact set such that $\overline{W^s(w)} = \mathrm{supp}(\mu)$ $\forall w \in X$. So, given $\varepsilon > 0$ there is a neighborhood V of X such that $W^s(z)$ is $\varepsilon-$dense in $\mathrm{supp}(\mu)$ if $z \in V \cap \mathrm{supp}(\mu)$.

Now we know that there is $x_{cu} \in W^s(x) \cap W_C^{cu}(p) \neq \emptyset$ and $\alpha(x_{cu}) \subset X$ $\forall x \in \mathrm{supp}(\mu)$. Then, there is an $N > 0$ such that $\forall n \geq N$ and $\forall x \in \mathrm{supp}(\mu)$ (see Figure 2)

$$\emptyset \neq f^{-n}(W^s(x) \cap W_C^{cu}(p)) = f^{-n}(W^s(x)) \cap f^{-n}(W_C^{cu}(p)) \subset f^{-n}(W^s(x)) \cap V.$$

Since we obtain that the previous intersection is nonempty simultaneously for all $x \in \mathrm{supp}(\mu)$ we have indeed that $W^s(x) \cap V \neq \emptyset$ $\forall x \in \mathrm{supp}(\mu)$. Thus, every strong stable manifold in $\mathrm{supp}(\mu)$ is $\varepsilon-$dense and since ε was arbitrarily chosen we have that $\mathrm{supp}(\mu)$ is minimal.

It only remains to show that $\mathrm{supp}(\mu)$ is the unique minimal set for $\mathcal{W}^s$. The argument is very similar to the previous one. Instead of X consider the set $Y = \mathcal{P}er(f) \cap W^c(p)$ (the set of all periodic points in $W^c(p)$ even if they are not in $\mathrm{supp}(\mu)$). Then, Y has the property

that $\text{supp}(\mu) \subset \overline{W^s(w)} \ \forall w \in Y$. Now, mutatis mutandis the previous argument gives that $\text{supp}(\mu) \subset \overline{W^s(x)} \ \forall x \in \mathbb{T}^n$. This implies that $\text{supp}(\mu)$ is the unique minimal set for $\mathcal{W}^s$. $\square$

This theorem yields to the following natural questions:

QUESTION 6.6. *Is* $\text{supp}(\mu) = \mathbb{T}^n$? *Similarly, are all absolutely partially hyperbolic diffeomorphisms homotopic to a hyperbolic linear automorphism of* $\mathbb{T}^3$ *transitive? What happens if f is accessible? (Recall that accessibility is abundant, see* [4])

7. SOME COMMENTS ON A KATOK'S CONJECTURE

This work started trying to answer, in the partially hyperbolic setting, a conjecture by A. Katok. He conjectures that a system with positive topological entropy h must have invariant measures whose entropies attain all values between 0 and h. This conjecture is true for hyperbolic systems and for $C^{1+\alpha}$ surface diffeomorphisms (see [11] for a proof).

In this section we want to do the following observation that applies to many systems. Suppose that π semi-conjugates $f : M \to M$ and $g : N \to N$ (M and N are compact metric spaces). That is, $\pi : M \to N$ is a continuous and surjective map such that $\pi \circ f = g \circ \pi$. Suppose that the entropy of the "fibers" is 0, i.e. $h(\pi^{-1}(x), f) = 0 \ \forall x \in N$. Suppose that g satisfies the Katok's conjecture then, it is also the case for f. This is direct consequence of Ledrappier-Walters variational principle [12]. Then, the results in this paper gives that if f is an absolutely partially hyperbolic diffeomorphism homotopic to a hyperbolic automorphism A of $\mathbb{T}^n$, with one-dimensional center and with quasi-isometric strong foliations we have that f satisfies the above mentioned Katok's conjecture.

THEOREM 7.1. *Let* $\gamma \in [0, h_{top}(f)]$. *Then, there exists an* $f-$*invariant measure* ν *such that* $h_\nu(f) = \gamma$.

Moreover, the same is true for the three dimensional partially hyperbolic diffeomorphisms with compact center leaves considered in [18].

REFERENCES

[1] R. Bowen, Markov partitions for Axiom A diffeomorphisms. *Amer. J. Math.* 92 (1970), 725–747.

[2] M. Brin, On dynamical coherence, *Ergodic Theory Dynam. Systems* 23 (2003), 395-ï¿401.

[3] M. Brin, D. Burago, S. Ivanov, Dynamical coherence of partially hyperbolic diffeomorphisms of the 3-torus, *J. Mod. Dyn.*, Vol. 3, 1–11, 2009.

[4] K. Burns, F. Rodriguez Hertz, M. A. Rodriguez Hertz, A. Talitskaya, R. Ures, Density of accessibility for partially hyperbolic diffeomorphisms with one-dimensional center. *Discrete Contin. Dyn. Syst.* 22 (2008), no. 1-2, 75–88.

[5] J. Buzzi, T. Fisher, M. Sambarino, C. Vasquez, Maximal Entropy Measures for certain Partially Hyperbolic, Derived from Anosov systems, to appear in *Ergodic Theory Dynam. Systems*.

[6] W. Cowieson, L.-S. Young, SRB measures as zero-noise limits, *Ergodic Theory Dynam. Systems* 25 (2005), no. 4, 1115–1138.

[7] L. J. Díaz, T. Fisher, Symbolic extensions for partially hyperbolic diffeomorphisms, *Discrete Contin. Dyn. Syst.* 29 (2011), no. 4, 1419–1441.

[8] J. Franks, Anosov diffeomorphisms. Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968) pp. 61–93 Amer. Math. Soc., Providence, R.I., 1970.

[9] A. Hammerlindl, Leaf conjugacies of the torus, to appear in *Ergodic Theory Dynam. Systems* and *Phd Thesis, University of Toronto*, 2009.

[10] Y. Hua, R. Saghin, Z. Xia, Topological entropy and partially hyperbolic diffeomorphisms. *Ergodic Theory Dynam. Systems* 28 (2008), no. 3, 843–862

[11] A. Katok, L. Mendoza, Dynamical systems with nonuniformly hyperbolic behavior, supplement to *Introduction to the Modern Theory of Dynamical Systems*, by A. Katok and B. Hasselblatt, Cambridge University Press, 1995.

[12] F. Ledrappier, P. Walters, A relativised variational principle for continuous transformations. *J. Lond. Math. Soc.* (2) 16 (1977), no.3, 568–576.

[13] R. Mañé, Contributions to the stability conjecture. *Topology*, 17 (1978) 383ï¿396.

[14] G.Margulis, On some aspects of the theory of Anosov systems, With a survey by Richard Sharp: Periodic orbits of hyperbolic flows, Springer Monographs in Mathe- matics. Springer-Verlag, 2004.

[15] M. Misiurewicz, Diffeomorphism without any measure with maximal entropy, *Bull. Acad. Pol. Acad. Sci. Math.* 21 (1973), 903–910.

[16] S. Newhouse, Continuity properties of entropy. *Ann. of Math.* (2) 129 (1989), no. 2, 215–235.

[17] S. Newhouse, L. S. Young, Dynamics of certain skew products, Geometric dynamics (Rio de Janeiro, 1981), 611–629, Lecture Notes in Math., 1007, Springer, Berlin, 1983.

[18] F. Rodriguez Hertz, M. Rodriguez Hertz, A. Tahzibi, R. Ures, Maximizing measures for partially hyperbolic systems with compact center leaves, to appear in *Ergodic Theory Dynam. Systems.*

[19] F. Rodriguez Hertz, M. A. Rodriguez Hertz, R. Ures, A non-dynamical coherent example in $\mathbb{T}^3$, preprint.

[20] M. Shub, Topologically transitive diffeomorphisms on $\mathbb{T}^4$. Lect. Notes on Math. 206 (1971), 39.

[21] B. Weiss, Intrinsically ergodic systems. *Bull. Amer. Math. Soc.* 76 (1970) 1266–1269.

IMERL-Facultad de Ingeniería, Universidad de la República, CC 30 Montevideo, Uruguay.

E-mail address: ures@fing.edu.uy

URL: http://www.fing.edu.uy/∼ures