On existence and uniqueness of weak foliations in dimension 3

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Abstract. In this paper we will prove some results about integrability of the weak invariant bundles for partially hyperbolic diffeomorphisms in dimension 3. We deal with the problems of existence and uniqueness in case we have transitivity and denseness of periodic orbits. We prove that if we have two crossing central curves contained in a weak-unstable disk then E^{cu} is uniquely integrable. We also obtain the integrability of the center bundle if the compact periodic center curves are dense.

1. Introduction

A diffeomorphism $f : M \to M$ is partially hyperbolic if TM splits into three invariant bundles such that two of them, the strong bundles, are hyperbolic (one is contracting and the other expanding) and the third, the center bundle, has an intermediate behavior (see Section 2 for a precise definition).

The integrability of the central distribution is one of the more striking problems in the study of partial hyperbolic systems. So, despite that our results are for 3-dimensional manifolds, let us start by giving a quick overview about the state of the art in the general problem (see also [**20**, Section 7]).

In general, given a plane field $E \subset TM$, there are two possible obstructions to the integrability of E:

(1) E does not satisfy the Froebenius bracket condition.
(2) E lacks enough differentiability.

In the partially hyperbolic setting, let us mention that although there are examples of non integrable central distributions (first observed by Wilkinson in [**23**]), in these examples the problem is the Froebenius bracket condition and not the differentiability.

In case E^c (or at least $E^{c\sigma}, \sigma = s, u$) is smooth enough the problem reduces to the Froebenius condition. In [**12**] (see also [**20**]) it is proven the integrability of the center distribution when some bunching condition is available. The bunching condition is a restriction to the strength of the contraction and expansion rates inside the center bundle. Namely, that the hyperbolicity of the strong bundles dominates the non-conformality of the center bundle. The result mentioned above suggests that the Froebenius part of the integrability problem is intimately related

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to bunching, that is, if f satisfies some bunching condition then E^{cs} and E^{cu} should be "involutive".

One can ask if the integrability condition can be weakened. For instance, if $\dim E^c = 1$ the Peano Existence Theorem gives us the existence of 1-dimensional manifolds tangent at every point to E^c. Related to this we want to mention the weak-integrability notion defined in [**7**]. A bundle $E \subset TM$ is said to be weakly integrable if for every point $x \in M$ there is a complete immersed manifold $x \in W(x)$ tangent to E. In [**7**], Brin, Burago and Ivanov prove that if the center bundle is one dimensional then E^{cs} and E^{cu} are weakly integrable. Also they prove that, for any $\dim E^c$, if two partially hyperbolic diffeomorphisms are homotopic through a path of partially hyperbolic diffeomorphisms and E^{cs} is weakly integrable for one of them then E^{cs} is also weakly integrable for the other. This naturally leads us to the following problem (see [**20**]):

Problem 1. *If f is center-bunched, is the center bundle weakly-integrable?.*

Other result under a different kind of hypothesis was obtained by Brin in [**6**]. A foliation $\mathcal{W}$ of a simply connected Riemannian manifold is said to be quasi-isometric if there are constants a and b such that whenever x and y are in the same leaf then $d_W(x, y) \leq ad(x, y) + b$, where $d_W(x, y)$ denotes the distance on the leaf W. In [**6**] Brin proved that if the unstable foliation $\mathcal{W}^u$ is quasi-isometric in the universal cover $\tilde{M}$, then the distribution E^{cs} is locally uniquely integrable. Of course if $\mathcal{W}^s$ is quasi-isometric in $\tilde{M}$ then E^{cu} is locally uniquely integrable, and if both are quasi-isometric in $\tilde{M}$ then E^c is locally uniquely integrable. Although the unstable foliation does not need to be quasi-isometric in the universal covering, an example is the geodesic flow on a hyperbolic surface, it is quasi-isometric in some interesting examples. Moreover, Burago-Ivanov proved that $\mathcal{W}^\sigma$, $\sigma = s, u$, is quasi-isometric if $M = \mathbb{T}^3$. We observe that the definition of partial hyperbolicity used in [**6**] (the same definition is given in [**7, 9**]) is more restrictive than the pointwise one used in this paper.

Under other viewpoint, it can be studied what happens to nearby diffeomorphisms when it is known that f has a center foliation. Given a partially hyperbolic diffeomorphism $f : M \to M$ having an invariant center foliation $\mathcal{F}^c$, we define an $\epsilon-$pseudo-orbit respecting the central plaques to be a sequence x_n, $n \in \mathbb{Z}$, such that $f(x_n) \in \mathcal{F}^c_\epsilon(x_{n+1})$. We say that f is plaque expansive at $\mathcal{F}^c$ if there is an $\epsilon > 0$ such that if x_n and y_n are $\epsilon-$pseudo-orbits preserving the central plaques and $d(x_n, y_n) < \epsilon$ for every $n \in \mathbb{Z}$ then $x_0 \in \mathcal{F}^c_\epsilon(y_0)$. The main reference for plaque expansivity is still [**15**].

If $\mathcal{F}^c$ is a C^1 foliation it is plaque expansive ([**15**]). Non-C^1 central foliations are known to be plaque expansive only in some cases.

Problem 2. *Are the central foliations always plaque expansive? What about when the strong foliations are quasi-isometric?*

One of the main consequences of plaque expansiveness is that the center bundle remains integrable when f is perturbed. Moreover,

Theorem ([**15**]). *Let $f : M \to M$ be a plaque expansive partially hyperbolic diffeomorphism. Then there is a neighborhood of f, U, such that if $g \in U$ then g leaves invariant a plaque expansive center foliation $\mathcal{F}^c_g$ and there is an homeomorphism $h : M \to M$ such that $h\left(\mathcal{F}_f(x)\right) = \mathcal{F}_g\left(h(x)\right)$ and $h\left(f\left(\mathcal{F}_f(x)\right)\right) = g\left(h\left(\mathcal{F}_f(x)\right)\right)$.*

In [**15**] the following question was posed:

Problem 3. *If f is partially hyperbolic and plaque expansive at $\mathcal{F}^c$, is $\mathcal{F}^c$ the unique $f-$invariant foliation tangent to E^c?*

Let us also mention that integrability, uniqueness and plaque expansiveness are guaranteed if the lengths of the curves tangent to the central bundle remain bounded under iteration. If the diffeomorphism is an isometry when restricted to E^c this result was announced in [**15**]. See [**20**, Theorem 7.5] for a proof.

1.1. Dimension 3. Although it is far from being solved, the panorama of the integrability problem seems to be clearer in the 3-dimensional case. We remark that in all known 3-dimensional examples the center bundle is uniquely integrable which implies the unique integrability of $E^{c\sigma}$, $\sigma = s, u$.

Firstly, let us mention the fundamental results obtained by Burago and Ivanov ([**9**]). They were able to prove that for any 3-dimensional partially hyperbolic diffeomorphisms there are branching foliations tangent to $E^{c\sigma}, \sigma = s, u$. By a branching foliation they mean, roughly speaking, a collection of immersed surfaces such that for each point passes (at least) one of these surfaces and the surfaces do not cross each other (they can be tangent and moreover coincide on large sets, see Definition 3.1). They have also proved that these branched foliations can be approximated by true foliations without Reeb components. In particular, this implies that there are no partially hyperbolic diffeomorphisms on $\mathbb{S}^3$.

Secondly, if f is transitive under the hypothesis of the existence of an invariant closed central curve γ with some especial semi-local condition on the intersection of $W^s_\delta(\gamma)$ and $W^u_\delta(\gamma)$ for some δ (not small) Bonatti and Wilkinson [**4**] proved the integrability of both weak bundles E^{cs} and E^{cu}. In fact, they gave a much more accurate description of the center foliation in these cases.

Finally, in [**19**] the authors proved the unique integrability of the central distribution when the diffeomorphism f is transitive and there are no periodic points at all.

1.2. Statement of the results. A foliation $\mathcal{F}$ stands for a partition of M where each element of the partition is a smooth immersed manifold $\mathcal{F}(x)$ and $T_x\mathcal{F}(x) \subset TM$ varies continuously with x. We say that a continuous plane field (distribution) E on TM is integrable if there is a foliation $\mathcal{F}$ such that $T_xF(x) = E(x)$ for every x in M. We say that E is uniquely integrable if there is only one foliation $\mathcal{F}$ such that $T_x\mathcal{F}(x) = E(x)$ for every x in M and in this case we shall call this foliation, the foliation tangent to E.

Let $\mathcal{I}^\sigma = \{f \in \mathcal{PH}; E^{c\sigma}_f \text{ is integrable}\}$, $\sigma = s, u$ and $\mathcal{I}^{su} = \mathcal{I}^s \cap \mathcal{I}^u$.

We will say that a distribution E has the *uniqueness property* if there exists at most one foliation tangent to E. Let $\mathcal{U}^\sigma = \{f \in \mathcal{PH}; E^{c\sigma}_f$ satisfies the uniqueness property$\}$, $\sigma = s, u$ and $\mathcal{U}^{su} = \mathcal{U}^s \cap \mathcal{U}^u$.

Finally, let $\mathcal{UI}^\sigma = \mathcal{U}^\sigma \cap \mathcal{I}^\sigma, \sigma = s, u, su$, i.e. the set of diffeomorphisms with, respectively, either E^{cs} or E^{cu} or both bundles uniquely integrable.

We remark that there are examples of continuous distributions not having the uniqueness property. In [**3**] Bonatti and Franks obtained a Hölder continuous line field of the plane tangent to more than one foliation.

Theorem 1.1. *Let $f \in \mathcal{PH}(M^3)$ be transitive and such that $\overline{\mathcal{P}er(f)} = M$, E^c is orientable and f preserves its orientation. Then, $f \in \mathcal{UI}^s \cup \mathcal{UI}^u \cup \mathcal{U}^{su}$.*

If f is accessible, and either E^c is non orientable or, E^c is orientable but f does not preserves its orientation then $f \in \mathcal{UI}^s \cup \mathcal{UI}^u$.

In other words either at least one of the weak bundles (E^{cs}, E^{cu}) is uniquely integrable or both of them have the uniqueness property.

Recall that f is accessible if for any pair of points $x, y \in M$ there exists a piecewise smooth path joining x with y consisting of a finite number of arcs such that each of them is contained either in a strong unstable manifold or in a strong stable manifold.

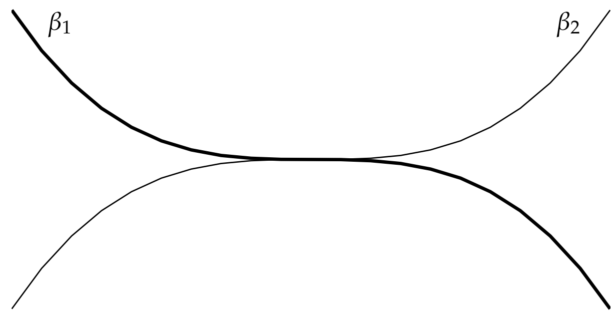

Figure 1. Two locally crossing curves in a center unstable leaf

We also obtain a local condition that implies integrability. Suppose that there are two central curves β_i, $i = 1, 2$, passing trough x that are contained in a local center unstable manifold $W^u_{loc}(\beta_1)$. Observe that β_1 separates $W^u_{loc}(\beta_1)$ into two connected components. We will say that β_i, $i = 1, 2$, are *locally crossing curves* if there are points of β_2 in each connected component of $W^u_{loc}(\beta_1) \setminus \beta_1$.

Theorem 1.2. *Let f be as in Theorem 1.1 and suppose that there exist two locally crossing curves contained in a local center unstable surface. Then, E^{cu} is uniquely integrable.*

We have to mention that the density of periodic points and accessibility are quite abundant once we have transitivity. Both of them are C^1-open and dense properties among the partially hyperbolic robustly transitive diffeomorphisms. The fact that C^1-generically periodic points are dense is a consequence of the C^1-Closing Lemma ([**17**]). In our context it can be proved that periodic points are dense for an open and dense subset of diffeomorphisms of the robustly transitive ones (see [**2**]). Openness and denseness of accessibility was proved in [**14**] for the C^1-topology, in [**18**] it was proved for the conservative case ($\Omega(f) = M$ is enough), $\dim(E^c) = 1$ and the C^r-topology, $r \geq 1$; and in [**11**] for the same setting, but without the conservative hypothesis. Moreover, in the conservative setting, there is a C^1-open and dense set of $C^{1+\text{Hölder}}-$diffeomorphisms with non-zero center Lyapunov exponent (just combine the results of [**1**] and [**10**]). Recall that non-zero Lyapunov exponents imply denseness of periodic points([**16**]). However, we observe that we do not require a dense set of hyperbolic periodic points (it would be a generic hypotheses anyway).

In many examples of partially hyperbolic diffeomorphisms we have that there exists a dense set of periodic closed curves tangent to E^c. Some of these examples are the time-one map of Anosov flows, skew products over Anosov diffeomorphisms, certain affine maps of nil-manifolds (see **[22]**). In fact, the only known examples where this is not the case, in dimension 3, are certain kind of DA-diffeomorphisms of $\mathbb{T}^3$. Our next theorem deals with this case. By a closed central curve we are meaning an embedding of S^1 tangent to E^c at every point .

Theorem 1.3. *Let $f \in \mathcal{PH}(M^3)$ with orientable E^c and such that the set of periodic closed central curves is dense in M. Then, $f \in \mathcal{I}^s \cap \mathcal{I}^u$.*

The proof of this theorem will be given in Section 7.

1.3. Some final comments. A partially hyperbolic diffeomorphism f is said to be *dynamically coherent* if there are invariant foliations $\mathcal{W}^{c\sigma}$, $\sigma = s, u$, tangent to $E^{c\sigma}$, $\sigma = s, u$ ($f \in \mathcal{I}^s \cap \mathcal{I}^u$). No uniqueness is required. In this case both foliations $\mathcal{W}^{c\sigma}$, $\sigma = s, u$, are sub-foliated by a foliation tangent of E^c and whose leaves are the intersections of the leaves of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$. When we obtain unique integrability of an invariant bundle then, invariance of the integral foliation is automatic. Hence, when we obtain unique integrability of both bundles $E^{c\sigma}$, $\sigma = s, u$, f is dynamically coherent. Since unique integrability of E^c implies unique integrability of both bundles $E^{c\sigma}$, $\sigma = s, u$ we arrive again to the conclusion that f is dynamically coherent if E^c is uniquely integrable.

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2. Preliminaries

Let M be a compact Riemannian manifold and denote by $\mathrm{Diff}^r(M)$ the set of C^r diffeomorphisms. In what follows we shall consider a *partially hyperbolic* $f \in \mathrm{Diff}^r(M)$, what means one admitting a non trivial Df-invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$, such that all unit vectors $v^\sigma \in E^\sigma_x$ with $\sigma = s, c, u$ and $x \in M$ verify:

$$\|Df(x)v^s\| < \|Df(x)v^c\| < \|Df(x)v^u\|$$

for some suitable Riemannian metric, which we call *adapted*. It is also required that the norm of the operators $Df(x)|_{E^s}$ and $Df^{-1}(x)|_{E^u}$ be strictly less than 1.

We shall denote $\mathcal{PH}^r(M)$ the family of C^r partially hyperbolic diffeomorphisms of M. From now on we will consider only the case $\dim M = 3$. Moreover, since the regularity of f (of course r must be ≥ 1) has no role in our results we will denote $\mathcal{PH}(M)$ for $\mathcal{PH}^1(M)$.

It is a known fact that, for $f \in \mathcal{PH}(M)$, there are foliations $\mathcal{W}^\sigma$ tangent to the distributions E^σ for $\sigma = s, u$ (see for instance **[8]**).

Due to the Peano Existence Theorem, for each $x \in M$ there are curves $\alpha_x(t)$ such that $\alpha_x(0) = x$ and $\dot\alpha_x(t) \in E^c(\alpha_x(t)) \setminus \{0\}$ for some open interval of parameters t containing 0. We shall call these curves *central curves through* x, and denote by $W^c_{loc}(x)$ the component of a central curve through x intersected by a small ball. It is easy to see that f takes central curves into central curves.

Denoting the leaf of $\mathcal{W}^\sigma$ through x by $W^\sigma(x)$, with $\sigma = s, u$, we write, as usual, $W^\sigma_{loc}(x)$ for the connected component of $W^\sigma(x) \cap B(x)$, where $B(x)$ is a small ball around x. Observe that for any choice of $W^c_{loc}(x)$, the sets

$$W^\sigma_{loc}(W^c_{loc}(x)) = \bigcup_{y \in W^c_{loc}(x)} W^\sigma_{loc}(y) \qquad \sigma = s, u$$

are C^1 (local) manifolds tangent to the bundle $E^{c\sigma} = E^\sigma \oplus E^c$ (with $\sigma = s, u$) at every point (see [**7**, Proposition 3.4]). For further use we will call, respectively, $W^{cs}_{loc}(x)$ and $W^{cu}_{loc}(x)$ the sets obtained as above depending, as it is obvious, on the choice of $W^c_{loc}(x)$. Moreover, as in the definition above let $W^\sigma_{loc}(A) = \bigcup_{x\in A} W^\sigma_{loc}(x)$.

Remark 2.1. *Let us note that, given $x, y \in M$, for all $W^{cs}_{loc}(x)$ such that $y \in W^{cs}_{loc}(x)$, there exists a central curve $W^c_{loc}(y)$ through y contained in $W^{cs}_{loc}(x)$ (see* [**7**]*).*

3. Branching foliations and integrability

Let us begin with some definitions. Following Burago-Ivanov ([**9**]) we say that a *surface* is a C^1−immersion $\phi : U \to M$ where U is a simply connected smooth 2-dimensional manifold. A surface $\phi : U \to M$ is *complete* if U has no boundary and the induced Riemannian metric on U is complete. We say that a curve $\gamma : I \to M$ (where I is an interval) lies on ϕ if γ can be represented as $\gamma = \phi \circ \gamma_0$ where γ_0 is a curve in U. We will abuse of notation and make no distinction between γ and γ_0.

A *neighborhood of* ϕ is an immersion $F : U \times \mathbb{R} \to M$ such that $F(x, 0) = \phi(x)$ for all $x \in U$. A curve $\gamma : I \to M$ *crosses* ϕ if there is an interval $J \subset I$ such that $\gamma|_J$ can be represented as $F \circ \gamma$ where F is a neighborhood of ϕ and $\tilde{\gamma} : J \to U \times \mathbb{R}$ is a curve which intersects both $U \times (0, +\infty)$ and $U \times (-\infty, 0)$. Clearly this definition does not depend on the choice of F.

Two surfaces ϕ_1 and ϕ_2 *topologically intersect* if there is a curve which lies on ϕ_1 and crosses ϕ_2. It is easy to see that this definition is symmetric with respect to ϕ_1 and ϕ_2.

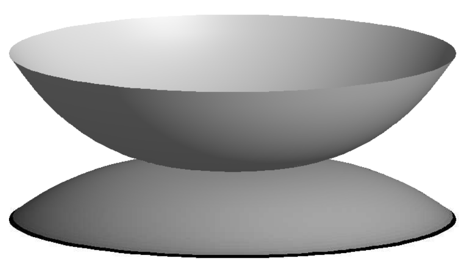

Figure 2. Leaves of a branching foliation

Definition 3.1. *A* branching foliation *in M is a collection of pairwise topologically non-intersecting complete surfaces whose images cover M.*

In [**9**], Burago and Ivanov proved that it is always possible to obtain a branching foliation tangent of $E^{c\sigma}$ for $\sigma = s, u$. In this work we will prove that if f is transitive and $\overline{\mathcal{P}er(f)} = M$ then the set of *all* complete surfaces tangent to at least one of the weak bundles form a branching foliation.

Definition 3.2. *We say that a distribution E is* almost integrable *if the set of all complete surfaces tangent to E form a branching foliation.*

In our context (E is one of the weak bundles of a partially hyperbolic diffeomorphism), if E is almost integrable when we intersect the branching foliation with a transverse surface at a branching point the situation is like in Figure 3.

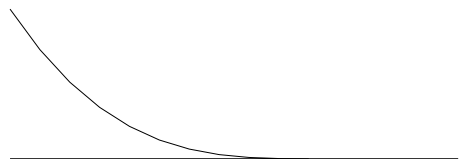

Figure 3. Allowed branching points

The proof of Proposition 3.4 of **[7]** shows that any disk tangent to E can be extended to a complete surface. Suppose that we have two surfaces intersecting "quadratically" with a transverse surface like in the left picture of Figure 4. Observe that in this case one can choose other surfaces tangent to E and topologically intersecting, like in the right picture of Figure 4.

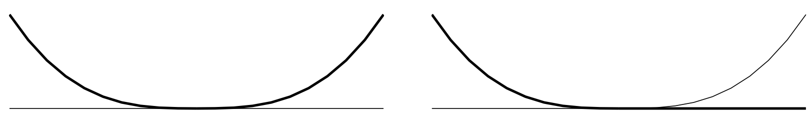

Figure 4. Not allowed branching points

Theorem 3.3. *Let $f \in \mathcal{PH}(M^3)$ be transitive and such that $\overline{\mathcal{P}er(f)} = M$, E^c is orientable and f preserves its orientation. Then there are two possibilities:*

(1) *Either E^{cs} or E^{cu} is uniquely integrable ($f \in \mathcal{UI}^s \cup \mathcal{UI}^u$).*
(2) *Both E^{cs} and E^{cu} are almost integrable.*

If E^c is non orientable or, if it is orientable but f does not preserves its orientation and, in addition, f is accessible then, either E^{cs} or E^{cu} is uniquely integrable ($f \in \mathcal{UI}^s \cup \mathcal{UI}^u$).

This theorem and the proposition below imply Theorem 1.1.

Proposition 3.4. *If E^{cu} is almost integrable then it has the uniqueness property ($f \in \mathcal{U}^u$).*

Proof of Proposition 3.4. Suppose that there exist two different foliations $\mathcal{W}$ and $\mathcal{W}'$ tangent to E^{cu}. Then there is a branching point x at which two different leaves $L_1 \in \mathcal{W}$ and $L_1' \in \mathcal{W}'$ meet (see Figure 5). If we take a sufficiently small stable segment joining L_1 with L_1', then any complete surface tangent to E^{cu} through a point in this stable segment will contain x, like the surface L_2 in Figure 5, or else, we would have that L_1 and L_2 or L_1' and L_2 are topologically intersecting surfaces. This would prevent $\mathcal{W}$ and $\mathcal{W}'$ from being a foliation. □

We postpone the proof of Theorem 3.3 until Section 5.

We want to pose some related problems. There are some weaker forms of partially hyperbolicity. We can allow, for instance, one of the strong bundles to be

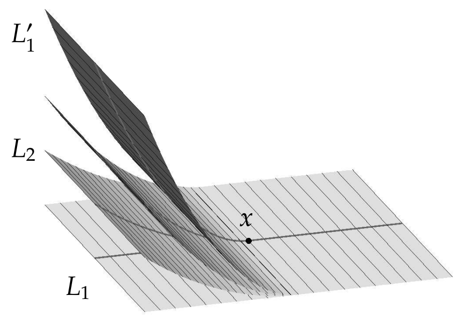

FIGURE 5. Proposition 3.4

trivial. In dimension 3, with this weaker definition, Díaz, Pujals and Ures ([**13**]) have proved that C^1-robustly transitive diffeomorphisms are partially hyperbolic. Moreover, if $TM = E^s \oplus E^{cu}$, they have proved that the Jacobian restricted to E^{cu} grows exponentially. In this case we say that E^{cu} is volume expanding. So, it is a natural question if Burago-Ivanov methods can be adapted to this setting.

PROBLEM 4. *If f admits an invariant splitting $TM = E^s \oplus E^{cu}$ where E^s is contracting and E^{cu} is volume expanding, is E^{cu} integrable?, or, can E^{cu} be approximated by integrable plane fields?*

The study of partially hyperbolic sets that are not the whole manifold has also proved useful to understand dynamics beyond hyperbolicity. For instance, it could be useful to give an answer to the following problem.

PROBLEM 5. *Is it possible to make a coherent construction like the one in the Burago-Ivanov paper*[**9**]*, when the partially hyperbolic set is not the whole manifold?*

Taking Burago-Ivanov result [**9**] into account, a positive answer to Problem 4 would give a negative answer to the problem below, at least in the C^1-topology.

PROBLEM 6. *Is there a robustly transitive diffeomorphism on S^3?*

4. Some auxiliary results

DEFINITION 4.1. *Let E be a continuous plane field of dimension k. A family $\mathcal{D}$ of $k-$disks tangent to E is* coherent *if for every pair of disks $D_i \in \mathcal{D}$ $(i = 1, 2)$ we have that $D_1 \cap D_2$ is a relatively open set in D_i, $i = 1, 2$.*

The following proposition, that has exactly the same proof of [**4**, Proposition 1.6] (see also Remark 1.10 therein), is an important tool in proving integrability of codimension 1 distributions.

PROPOSITION 4.2. *Let M be a compact n-manifold, and let E be a continuous codimension 1 plane field (distribution of hyperplanes) on M. Assume that there exists a coherent family $\mathcal{D}$ of disks for E such that:*

- *The disks of $\mathcal{D}$ have fixed radii.*

- *The center of the disks form a dense subset of M.*

Then there is a unique continuous foliation $\mathcal{F}$ whose leaves are C^1 and are tangent to E, and is such that each disk in $\mathcal{D}$ is contained in a leaf of $\mathcal{F}$.

Moreover, the same proof gives that E is locally uniquely integrable if, in addition, we have that every disk D of $\mathcal{D}$ is locally unique. By D being locally unique we mean that for every $y \in D$ and a disk tangent to E, $(y \in)\tilde{D}$, we have that $D \cap \tilde{D}$ is a neighborhood of y in D.

We say that E is locally uniquely integrable if it is integrable (let $\mathcal{F}$ be a foliation tangent to E) and for any curve γ such that $\dot{\gamma} \in E$ we have that γ is inside one leaf of $\mathcal{F}$. Obviously, local uniqueness of integrability implies uniqueness. For simplicity of exposition, along the paper we will talk about unique integrability but, in fact, in all cases when we obtain unique integrability we obtain it locally.

The following lemma is proved in [**19**, Remark 3.7]. It has proved useful to obtain unique integrability of the center bundle (see [**19, 21**]). It says that at points at which there is no local uniqueness of the central bundle the central curves should grow (at least) up to a constant that does not depend on the point.

Lemma 4.3. *Let E^c be 1-dimensional and $\delta > 0$ small enough. Suppose that E^c is non uniquely integrable at x when restricted to some $W^{cu}_{loc}(x)$. Then, for each connected central subsegment c of extreme x contained in the adequate component of $W^{cu}_{loc}(x) \setminus W^{u}_{loc}(x)$, there is $N > 0$ for which $f^n(c) \not\subset B_\delta(f^n(x))$ for all $n \geq N$.*

The adequate component of $W^{cu}_{loc}(x) \setminus W^{u}_{loc}(x)$ is the one that contains two different central curves through x.

5. Classification of periodic points and proof of Theorem 3.3

The main goal of this section is to prove Theorem 3.3. To this end we will give a classification of periodic points according to the behavior of the endpoints of a maximal central curve containing the periodic point such that it is uniquely tangent to E^c at every point of it.

5.1. Classification of periodic points. We shall assume here that the bundle E^c is oriented and that f preserves its orientation. In Section 5.3 we discuss what happens when this is not the case.

Let p be a periodic point and let γ_p be the maximal center curve containing p such that it is locally the unique tangent curve at all of its points. Note that γ_p could contain its endpoints or not, it can be an infinite line, a semi-line, a closed curve or even a point.

More precisely, $\gamma_p = \{p\}$ if for any center curve c with endpoint p there exists a point $y \in c$ and a center curve $\hat{c}$ such that $c \cap \hat{c}$ is not open in c. In other words local uniqueness must fail at both sides of p. See Figure 6.

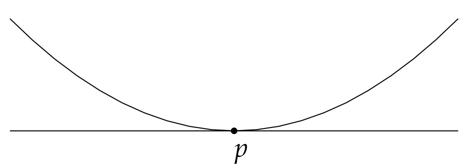

Figure 6. $\gamma_p = \{p\}$

γ_p is an infinite line if p is contained in an infinite center line which is locally unique at each point. γ_p is a closed curve if p is contained in a closed center curve which is locally unique at each point. Endpoints of γ_p are defined by maximality. The endpoint b is not included in γ_p if there exists a center curve $\tilde{\gamma}$ beginning at b and going in the same direction as γ_p such that $\gamma_p \cap \tilde{\gamma} = \{b\}$. Otherwise $b \in \gamma_p$. See Figure 7.

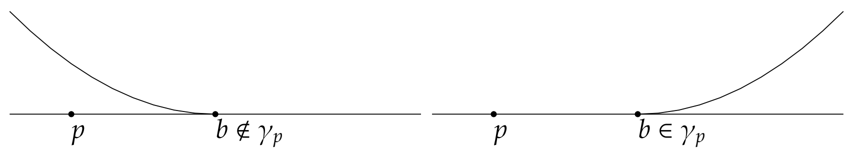

FIGURE 7. Behavior of endpoints

Observe that the maximality and the f-invariance of the properties of the endpoints imply that $f(\gamma_p) = \gamma_{f(p)}$. This implies that the endpoints are periodic.

Suppose that b is an endpoint such that $b \in \gamma_p$. Maximality implies that no prolongation of γ_p is locally uniquely integrable. Let β be a center curve with endpoint b and such that γ_p and β have opposite directions. That is, $\gamma_p \cup \beta$ contains b in its interior. Then, there exist $x \in \beta$ and a center curve $\tilde{\beta}$ such that x is an endpoint of $\tilde{\beta}$ and $\beta \cap \tilde{\beta} = \{x\}$. By taking intersections between $W^{c\sigma}_{loc}(\tilde{\beta})$, $\sigma = u, s$, and $W^{c\tau}_{loc}(\beta)$, $\tau = u, s$, we can suppose that $\tilde{\beta}$ is contained in $W^{cu}(\beta)$ (or in $W^{cs}(\beta)$). By Lemma 4.3 this implies that there is $N > 0$ such that the length of $f^n(\beta)$ is greater than δ for all $n \geq N$. In other words, f expands center curves beginning at b that are on the side of β. Moreover, it is not difficult to see that this implies that E^{cu} is uniquely integrable at $W^u_{loc}(\beta)$. That is, given $y \in W^u_{loc}(\beta)$ and an open surface W, $y \in W$ tangent to E^{cu} then $W^u_{loc}(\beta) \cap W$ is relatively open in $W^u_{loc}(\beta)$. Details may be found in [**19**].

Suppose, on the contrary, that $b \notin \gamma_p$. Then there is a periodic point $q \in \gamma_p$ such that there are no periodic points in (q, b) (assume, without loss of generality, that b is the right endpoint of γ_p). We are denoting by (q, b) the sub-arc of γ_p that has endpoints q and b and does not contain q nor b. $[q, b]$ is $(q, b) \cup \{q, b\}$ and $[q, b)$ is $(q, b) \cup \{q\}$, etc. Applying Lemma 4.3 again we have that the length of (q, b) is greater than δ. Moreover, q must attract or repel all points in $[q, b)$. In the attracting case E^{cs} is uniquely integrable at $W^s_{loc}([q, b))$ and in the repelling case E^{cu} is uniquely integrable at $W^u_{loc}([q, b))$.

We now classify the periodic points p of $f \in \mathcal{PH}(M^3)$ into five classes Λ_i, $i = 1, \ldots, 5$.

(1) Λ_1 is the set of periodic points p such that γ_p is a closed curve or an infinite line.
(2) Λ_2 is the set of periodic points p such that γ_p has at least one endpoint, and each endpoint b satisfies:
    (a) if $b \in \gamma_p$, and k is the period of p, then b attracts, under the action of f^k, the points in any center curve beginning at b and going to the

opposite direction of γ_p. That is γ_p is an attractor restricted to any center curve a little bit longer than γ_p.
   (b) if $b \notin \gamma_p$, and q is the periodic point in γ_p such that there are no other periodic point in $[q, b)$, then q attracts the points of $[q, b)$.

(3) Λ_3 is defined as Λ_2, but requiring that b, resp q, repel instead of attract. That is Λ_3 is Λ_2 for f^{-1}.
(4) Λ_4 is the set of periodic points p such that γ_p has two endpoints: the left one (according the orientation of E^c) satisfies the conditions of the endpoints of Λ_2 and the right one satisfies the conditions of Λ_3.
(5) Λ_5 is defined as Λ_4 but interchanging the role of left and right endpoints. That is Λ_5 is Λ_4 for f^{-1}.

5.2. Proof of Theorem 3.3: the orientable case. Assume that f preserves the orientation of E^c. Then, each set Λ_i is f-invariant. Moreover, the union of the sets Λ_i is the set of periodic points of f which (by hypothesis) is dense in M. Hence the closure of some Λ_i has non-empty interior. Thus, by the transitivity of f, such a Λ_i is dense in M. Then, we have five possibilities.

i) Λ_1 *is dense.* Take $p \in \Lambda_1$. Through p passes a center curve γ of length ν (ν does not depend on p) that is the unique curve tangent to E^c at all its points. Then, $W^s_{loc}(\gamma)$ and $W^u_{loc}(\gamma)$ also have this uniqueness property with respect to $E^{c\sigma}$, $\sigma = s, u$. Hence we have disks of uniform radius in $W^\sigma_{loc}(\gamma)$, $\sigma = s, u$, centered at p. Since Λ_1 is dense, Theorem 4.2 gives us that both weak bundles are uniquely integrable.

ii) Λ_2 *is dense.* Take $W^s_{loc}(\gamma_p)$ for $p \in \Lambda_2$. Let b_r (b_l) be the right (left) endpoint of γ_p. Suppose that no endpoints belong to γ_p. Then the lengths of (p, b_r) and (b_l, p) are greater than the constant δ given by Lemma 4.3. Since (b_l, b_r) is the unique curve tangent to E^c at all its points we have that there is a disk centered at p of radius depending on δ and contained in $W^s_{loc}(b_l, b_r)$ that has again the uniqueness property. Suppose now that some endpoint, say for instance b_r, belongs to γ_p. The remarks of the section above and the definition of Λ_2 imply that we can attach to $W^s_{loc}(\gamma_p)$ (that uniquely integrates E^{cs}) a surface $W^s_{loc}(\overline{\beta})$. This surface has the mentioned uniqueness property with respect to E^{cs} and $\overline{\beta}$ is a center curve with length that is again greater than δ. Then, we also obtain a center-stable disk centered at p of size depending only on δ and with the desired uniqueness. Finally, the density of Λ_2 and Theorem 4.2 imply unique integrability of E^{cs}.

iii) Λ_3 *is dense.* The proof above applied to f^{-1} implies that E^{cu} is uniquely integrable.

iv) Λ_4 *is dense.* The case where Λ_4 (or Λ_5), and none of the others, is dense is the case where we are not able to prove the integrability of at least one of the weak bundles. Instead, we obtain almost integrability. First of all, considerations as above imply that any point in Λ_4 has a semi-disk of unique integrability of E^{cu} with uniform size on its right side and a semi-disk of unique integrability of E^{cs} with uniform size on its left side. The radii of these semi-disks only depend on the constant δ given by Lemma 4.3.

Suppose that there exist two small center curves γ_i, $i = 1, 2$, such that $\gamma_2 \subset W^u_{loc}(\gamma_1)$ and $x \in \gamma_i$ for $i = 1, 2$. We claim that $\gamma_2 \subset \gamma_1$. If this were not the case there would exist a triangle-like set Δ formed formed by an arc in γ_1, an arc in γ_2 and a small unstable segment. Since Λ_4 is dense we can take $p \in \Lambda_4$ very close to the vertex of Δ that belongs simultaneously to both curves γ_i, $i = 1, 2$ and such that $W^s_\varepsilon(p) \cap \text{int}(\Delta) \neq \emptyset$ for ε very small. Then any center stable semi-disk on the left side of p, not too small, would have a point where it would not be the unique surface integrating E^{cs}. This contradicts our remark about the semi-disks of uniform size. This means that center curves inside weak unstable manifolds are unique when we flow to the right and they can bifurcate only when we flow to the left. Finally, this clearly implies that complete surfaces tangent to E^{cs} cannot topologically intersect. A symmetric reasoning shows that complete surfaces tangent to E^{cu} cannot topologically intersect either. This implies that both weak bundles are almost integrable (recall that given x there always exists a complete surface through x tangent to $E^{c\sigma}$, $\sigma = s, u$).

Suppose now that f does not preserve the orientation of E^c. In this case we have as an additional hypothesis that f is accessible. Since $\Omega(f^2) = M$, Brin has proved (see [**5**]) that f^2 is transitive too. Then we can apply our previous consideration to f^2. For the three first cases we arrive to the same conclusion. In case iv) we have that if $\Lambda_4(f^2)$ is dense iterating once by f we have that $\Lambda_5(f^2) = f(\Lambda_4)$. The considerations above imply that both weak bundles are uniquely integrable in this case.

5.3. Non orientable case. By taking a double covering $\hat{M}$ of M and lifting f we obtain a partially hyperbolic diffeomorphism $\hat{f} : \hat{M} \to \hat{M}$ with orientable central bundle. Since $\hat{M}$ is connected and f has the accessibility property we obtain that $\hat{f}$ is also accessible. Transitivity of f implies that $\Omega(\hat{f}) = \hat{M}$ so, $\hat{f}$ is transitive. If $\hat{f}$ does not preserve the central orientation, the result at the end of the subsection above implies the thesis. Observe that since we obtain *local* unique integrability, the foliations are also unique in M.

If $\hat{f}$ preserves the orientation of the central bundle we only have problems in case $\Lambda_4(\hat{f})$ (or $\Lambda_5(\hat{f})$) is dense, and none of the other $\Lambda_i(\hat{f})$ is. Take a look at the local situation. Let U be a small neighborhood in M and let $\hat{U}_i$, $i = 1, 2$, be its preimages under the covering projection. We can orientate E^c inside U so that the lift of this orientation coincide with the orientation of the center bundle of $\hat{f}$ inside one of the preimages, say U_1, and so that it does not coincide in the other one, U_2. Since $\Lambda_4(\hat{f})$ is dense we have a dense set of periodic points (in the whole manifold $\hat{M}$ and, in particular, in U_i, $i = 1, 2$) with semi-disks of uniform size uniquely tangent to the center stable bundle and located on the left side (similar for the center unstable bundle). Projecting onto U we have, with the local orientation of E^c that we have chosen, a dense set of periodic points with its semi-disks on its right side and a dense set with them on its left side. The arguments at the end of the section above imply the unique integrability in U which implies unique integrability.

6. Proof of Theorem 1.2

Theorem 1.2 is essentially proved in the previous section. Clearly Λ_1 and Λ_3 cannot be dense because E^{cu} is not uniquely integrable. The density of Λ_2 gives us the thesis. We can argue in a similar way and the remaining case is when E^c

is orientable, f preserves the central orientation and Λ_4 (or Λ_5) is dense. But the presence of crossing central curves implies that they are not unique neither when we flow to the right nor when we flow to the left. Then, density of Λ_4 (or Λ_5) is forbidden in this case and the theorem is proved.

7. Denseness of periodic closed curves

This section is devoted to proving Theorem 1.3.

First of all, let us observe that if γ is an embedded periodic center circle (periodic circles for short) the stable (unstable) manifold of γ is an injective immersion of either a cylinder or a Mœbius band (see, for instance, [**4**, Lemma 1.4]). For instance, if γ is $f-$invariant, transitivity would imply that these manifolds are dense. If they were complete we would have integrability of the weak bundles by Proposition 4.2. But whether these manifolds are complete is an open question.

Secondly, since periodic circles are dense we may take, for instance, the local stable manifolds of uniform size at each periodic circle. In fact, since periodic circles are normally hyperbolic we have these well defined stable and unstable manifolds. Then we can take a disk of uniform size inside the corresponding stable manifold through each point of every periodic circle. After this we can try to apply Proposition 4.2 again. We will show that this family of disks is coherent. We will need the following lemma.

Lemma 7.1. *Let γ be a periodic circle. Suppose that unique integrability fails at $x \in \gamma$ that is, there exists a center curve β such that $\beta \cap \gamma = \{x\}$. Then, x is periodic.*

Proof. If x is not periodic then it has infinite many different points of its orbit in γ. Suppose that E^c is not uniquely integrable at a non periodic point $x \in \gamma$. Then, it is not uniquely integrable at $f^{nk}(x)$, $\forall n \in \mathbb{Z}$. Take $l \in \mathbb{N}$ such that $l\delta$ is greater than the length of γ, being δ the constant of Lemma 4.3. We can choose l pairwise disjoint intervals of γ containing l different iterates of x. Hence, if we take a large enough iterate of f^k, Lemma 4.3 implies that γ contains l disjoint intervals of length greater than δ. This contradiction proves the lemma. □

Suppose that the family of center stable disks of uniform size mentioned above is not coherent. Then, there are at least two disks (with center in different periodic circles γ_1, γ_2) that intersect in a relatively non open set. Hence, the center foliation is not unique along a strong stable curve contained in both disks. Lemma 7.1 implies that this stable curve intersect both γ_1 and γ_2 at periodic points p_1 and p_2. Moreover, since p_1 and p_2 are in the same stable curve they coincide, $p := p_1 = p_2$. This is not a contradiction yet because we did not ask for disjointness of the periodic circles.

If we consider $W^s_{loc}(\gamma_i)$, $i = 1, 2$, at p we have an open set U (bounded by $W^s_{loc}(\gamma_i)$, $i = 1, 2,$) such that any center curve intersecting U intersects $W^s_{loc}(\gamma_1)$ or $W^s_{loc}(\gamma_2)$. If we take a periodic circle γ intersecting U the same argument given for γ_i, $i = 1, 2$, implies that the intersection of γ with, for instance, $W^s_{loc}(\gamma_1)$ is in γ_1 (and it is a periodic point). This contradicts the density of the periodic circles and proves Theorem 1.3.

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