ACCESSIBILITY AND STABLE ERGODICITY FOR PARTIALLY HYPERBOLIC DIFFEOMORPHISMS WITH 1D-CENTER BUNDLE

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Abstract. We prove that stable ergodicity is C^r open and dense among conservative partially hyperbolic diffeomorphisms with one-dimensional center bundle, for all $r \in [2, \infty]$.

The proof follows the Pugh-Shub program [29]: among conservative partially hyperbolic diffeomorphisms with one-dimensional center bundle, accessibility is C^r open and dense, and essential accessibility implies ergodicity.

1. Introduction

In the second half of the 19th century Boltzmann introduced the term *ergodic* within the context of the study of gas particles. Since then, even though in its initial formulation the Ergodic Hypothesis was ambiguous, ergodic theory grew up to be a useful tool in many branches of mathematics and physics.

Subsequent reformulations and developments turned the original ergodic hypothesis into the statement: *time average equals space average* for typical orbits, that is

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \phi(f^k(x)) = \int_M \phi \, d\mu \qquad \mu - \text{a.e.}x$$

A system is μ-*ergodic* if it satisfies the hypothesis above for all C^0 observables ϕ, or equivalently, if only full or null μ-volume sets are invariant under the dynamics. Around 1930, right after the first ergodic theorems appeared - [23], [3], [4] - it was conjectured that most conservative systems were ergodic.

With the Kolmogorov-Arnold-Moser (KAM) phenomenon (1954) it came out that there were full open sets of conservative non-ergodic systems [21]. Indeed, KAM theory presents, for small perturbations of integrable systems (elliptic dynamics), positive volume sets of invariant tori, which prevents ergodicity. These are examples of a *stably non ergodic* system.

On the other end of the spectrum, the work of Hopf [19], and later Anosov-Sinai [1, 2], gave full open sets of ergodic systems, a fact that was unknown up to

2000 *Mathematics Subject Classification.* Primary: 37D30, Secondary: 37A25.

This work was partially supported by FCE 9021, CONICYT-PDT 29/220 and CONICYT-PDT 54/18 grants.

that time. Anosov systems, also called completely hyperbolic dynamics, were for some time the only known examples of *stably ergodic* systems. By stably ergodic is meant a diffeomorphism in the interior of the set of the ergodic ones.

Almost three decades later, Grayson, Pugh and Shub got the first non-hyperbolic example of a stably ergodic system [17]. These examples have a *partially hyperbolic dynamics* [8], [18]: there are strong contracting and strong expanding invariant directions, but a center direction also appears. Since then, the area became quite active and many stably ergodic examples appeared, see [30] for a survey. Let us also mention that there are already examples of conservative stably ergodic systems that are *not* partially hyperbolic [35].

In this new context, Pugh and Shub have proposed the following:

Conjecture 1 *Stable ergodicity is C^r dense among volume preserving partially hyperbolic diffeomorphisms, for all $r \geq 2$.*

As far as we know, the conjecture above was first stated in 1995, at the International conference on dynamical systems held in Montevideo, Uruguay [27]. We thank Keith Burns for this information.

In this paper, we prove this conjecture is true in case the center bundle is one dimensional:

Theorem (Main). *Stable ergodicity is C^r dense among volume preserving partially hyperbolic diffeomorphisms with one dimensional center distribution, for all $r \geq 2$.*

In [29], Pugh and Shub proposed a program for the proof of this conjecture. This approach was based on the notion of *accessibility*: A diffeomorphism f has the *accessibility property* if the only non void set consisting of whole stable leaves and whole unstable leaves is the manifold M itself. It has the *essential accessibility property* if every measurable set consisting of whole stable leaves and whole unstable leaves has full or null volume. Clearly, accessibility implies essential accessibility. When talking about stable and unstable leaves we are referring to the leaves of the unique foliations tangent to the contracting and expanding directions, respectively.

Pugh and Shub suggested the following conjectures:

Conjecture 2: *Stable accessibility is dense among C^r partially hyperbolic diffeomorphisms, volume preserving or not, $r \geq 2$.*

In the case $\dim E^c = 1$, the accessibility property is always stable [15]. For the sake of simplicity, let us denote by $\mathrm{PH}^r_m(M)$ the set of partially hyperbolic C^r diffeomorphisms of M, preserving a smooth probability measure m. In this paper, we prove:

Theorem A. *Accessibility is open and dense in $\mathrm{PH}^r_m(M)$, for all $1 \leq r \leq \infty$, if the center distribution is one dimensional.*

In fact, we obtain that accessibility is C^1 open and C^∞ dense. Observe that the conjecture is established here only for the conservative case. The conjecture is settled in the general non conservative case also for center dimension one in [10] by extending the technics in this work.

Let us make some comments about the history of Conjecture 2. Anosov diffeomorphism are easily shown to have the accessibility property. In [33] Sacksteder gave the first example of an accessible non Anosov diffeomorphism. This was an affine diffeomorphism of a 3 dimensional nilmanifold. Brin and Pesin proved in [8] that the time one map of the k-frame flow over a surface of negative curvature is accessible and proved a theorem about the stable accessibility for some partially hyperbolic systems [8, Theorem 4.1.]. Grayson, Pugh and Shub proved in [17] that the time one map of the geodesic flow over a surface of negative curvature is *stably* ergodic. Later, Wilkinson extended this to the variable curvature case in [36]; then Katok and Kononenko extended this to the case of the time one map of a contact Anosov flow in [20]; and Burns, Pugh and Wilkinson proved in [9] stable accessibility for mixing Anosov flows. In [24], Niţică and A. Török proved stable accessibility is C^r dense for one-dimensional center bundle, under certain hypotheses (for instance, dynamical coherence and compact center leaves). Didier has proven in [15] that accessibility is an open property when the central dimension is 1. For any central dimension, Pugh and Shub proved in [28] that accessibility is stable whenever the strong bundles are smooth. Also, Burns and Wilkinson proved C^r density of stable accessibility among skew products in [11]. Shub and Wilkinson showed in [34] that ergodic linear automorphisms on tori can be C^r perturbed to become stably accessible and the first author proved in [32] that some of them are stably essentially accessible when the central dimension is 2. Finally, in [16], stable accessibility is shown by Dolgopyat and Wilkinson to be dense in the C^1 topology with no assumption on the dimension of the center bundle.

The second conjecture of the Pugh-Shub program is:

Conjecture 3: *Essential accessibility implies ergodicity among C^2 volume preserving partially hyperbolic diffeomorphisms.*

We also prove this conjecture in case the center dimension is one.

Theorem B. *Essential accessibility implies Kolmogorov (in particular, ergodicity) in $\mathrm{PH}^2_m(M)$, if the center distribution is one-dimensional.*

Let us mention that K. Burns and A. Wilkinson have recently presented a proof of Theorem B in [13, Corollary 0.2]. In fact, they deduce Theorem B from a more general theorem involving a technical condition named center bunching that is trivially satisfied when the dimension of the center bundle is 1 (see the beginning of section 4 for detailed definitions). Let us state their theorem:

THEOREM 1. [13, Theorem 0.1] *Let f be C^2, volume-preserving, partially hyperbolic and center bunched. If f is essentially accessible, then f is ergodic, and in fact has the Kolmogorov property.*

Let us mention a little bit of the history of the proof of Theorem B. The first attempt to prove Conjecture 3 appeared in [17], where M. Grayson, C. Pugh and M. Shub also proved the stable ergodicity of the time one map of the geodesic flows over surfaces of constant negative curvature. In this case, the center bunching condition was global over M in contrast to the point-wise condition in Theorem 1. They also needed another hypothesis called dynamical coherence which essentially means that the center-stable and center-unstable bundles are integrable. Subsequently, in the papers [36, 28, 29] the center bunching condition was improved, still in the global setting while the hypothesis of dynamical coherence was not touched at all. In [12] K. Burns and A.Wilkinson jumped from the global center-bunching condition to a point-wise, improved one, with the gain that now the condition is trivially satisfied when the central dimension is 1. But the dynamical coherence was still needed. Finally, in [13] they removed the dynamical coherence condition using the notion of fake foliations. The fake foliations consist, roughly speaking, of families of local foliations which are almost invariant and almost tangent to the invariant spaces.

In our case, when the central foliation is one dimensional, we were able to remove the dynamical coherence condition in [12] in a different manner. Instead of using fake disks, we use true integral curves of the center bundle. This integral curves are much easier to handle since they are everywhere tangent to the central bundle. We found this way of removing the dynamical coherence condition (when $\dim E^c = 1$) independently and simultaneously with [13]. So, for completeness, and because the proof in our case is a little bit simpler than in [13] we decided to include it here. See proof of Theorem B.

Let us mention that in [13] they also prove that differentiability condition in Theorem B can be improved to $C^{1+\text{Hölder}}$. We thank A. Wilkinson for this information.

Acknowledgements. We want to thank M. Shub for his support in a difficult moment. We also want to thank K. Burns for reading early versions of this manuscript and for useful remarks. We are also grateful to C. Pugh for many valuable suggestions. Also we would like to thank the two referees for all the corrections and comments.

2. PRELIMINARIES, NOTATION AND SKETCH OF THE PROOF

Let M be a compact Riemannian manifold, and m be a smooth probability measure on M. Denote by $\mathrm{Diff}^r_m(M)$ the set of C^r volume preserving diffeomorphisms. In what follows we shall consider a *partially hyperbolic* $f \in \mathrm{Diff}^r_m(M)$, that is, a diffeomorphism admitting a non trivial Df-invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$, such that all unit vectors $v^\sigma \in E^\sigma_x$ $(\sigma = s, c, u)$ with $x \in M$ satisfy:

$$\|T_x f v^s\| < \|T_x f v^c\| < \|T_x f v^u\|$$

for some suitable Riemannian metric. It is also required that $\|Tf|_{E^s}\| < 1$ and $\|Tf^{-1}|_{E^u}\| < 1$. We shall denote by $\mathrm{PH}^r_m(M)$ the family of C^r volume preserving partially hyperbolic diffeomorphisms of M.

It is a known fact that there are foliations $\mathcal{W}^\sigma$ tangent to the distributions E^σ for $\sigma = s, u$ (see for instance [8]). A set X will be called *σ-saturated* if it is a union of leaves of $\mathcal{W}^\sigma$, $\sigma = s, u$.

In this paper we will consider the case $\dim E^c = 1$. Due to the existence of solutions of differential equations with continuous vector fields, we can find small curves passing through each $x \in M$ that have everywhere nonzero tangent vector and are everywhere tangent to the bundle E^c. We shall call these curves *center curves through* x, and denote them by $W^c_{loc}(x)$ in order to distinguish them from the true foliations $\mathcal{W}^\sigma$, $\sigma = s, u$, since a priori they are not the unique integral curves tangent to E^c. It is easy to see that f takes center curves into center curves.

We shall denote by $\mathcal{W}^\sigma(x)$ the leaf of $\mathcal{W}^\sigma$ through x for $(\sigma = s, u)$ and will write $\mathcal{W}^\sigma_{loc}(x)$ for a small disk in $\mathcal{W}^\sigma(x)$ centered in x. For any choice of $W^c_{loc}(x)$, the sets

$$W^{\sigma c}_{loc}(x) = \mathcal{W}^\sigma_{loc}(W^c_{loc}(x)) = \bigcup_{y \in W^c_{loc}(x)} \mathcal{W}^\sigma_{loc}(y) \qquad \sigma = s, u$$

are C^1 (local) manifolds everywhere tangent to the sub-bundles $E^\sigma \oplus E^c$ for $\sigma = s, u$ (see, for instance [7, Proposition 3.4.]). The sets above depend on the choice of $W^c_{loc}(x)$.

Remark 2.1. *Observe that for all choices of $W^{sc}_{loc}(x)$ and $y \in W^{sc}_{loc}(x)$, there exists a center curve $W^c_{loc}(y)$ through y contained in $W^{sc}_{loc}(x)$ (see [7]).*

Observe also the following key property of the central manifolds that is a consequence of the continuity and transversality of the invariant bundles,

Lemma 2.1. *For each small $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(z, y) < \delta$ then $\mathcal{W}^s_\varepsilon(W^c_{loc}(y)) \cap \mathcal{W}^u_\varepsilon(z) \neq \emptyset$ or ,what is equivalent, $W^c_{loc}(y) \cap \mathcal{W}^s_\varepsilon(\mathcal{W}^u_\varepsilon(z)) \neq \emptyset$ for any choice of center curve through y.*

2.1. **Proof of Theorem A.** Let us say that a set Γ is *σ-saturated* if Γ is a union of leaves of $\mathcal{W}^\sigma$, $\sigma = s, u$. For the proof of Theorem A, we will see that C^r-generically, the *accessibility class* of a point x, that is, the minimal s- and u- saturated set that contains x, is the whole of M if the center bundle is one-dimensional. This property is known as the *accessibility property* and is open in $\mathrm{PH}^1_m(M)$ if the center bundle is one-dimensional [15].

The proof focuses on the open accessibility classes, and the first step is showing that for any periodic point, a perturbation can be made so that its accessibility class becomes open (Unweaving Lemma). Secondly, we obtain periodic points for any dynamics in $\mathrm{PH}^r_m(M)$ having non trivial open accessibility classes that do not cover M. A Kupka-Smale type genericity argument allows us to conclude:

PROPOSITION A.1. *C^r-generically in $\mathrm{PH}^r_m(M)$, $r \geq 2$, either one of the following properties holds:*

(1) *f has the accessibility property or*
(2) *$\mathrm{Per}(f) = \emptyset$ and the distribution $E^s \oplus E^u$ is integrable*

One would expect the second possibility is quite unstable under perturbations and, indeed, this is the case:

PROPOSITION A.2. *Situation (2) described above is meager in $\mathrm{PH}^r_m(M)$.*

We show that the Unweaving Lemma mentioned above also holds for non recurrent points. In this way, integrability of $E^s \oplus E^u$ can be broken by small perturbations.

In both cases, to have some control on how perturbations affect local invariant manifolds, we need the existence of points whose orbits keep away from the support of the perturbation (Keepaway Lemma A.4.2).

The two statements together imply Theorem A. This part is developed in §3.

2.2. **Proof of Theorem B.** For the proof of Theorem B, we shall mainly follow the lines in [17], [29] and [12]. This theorem was obtained independently of [13], though Burns and Wilkinson's result is more general. We decided to include Theorem B here for completeness, and because our proof is simpler in the sense that it uses true integral curves instead of fake foliations, which are a difficult technical step (see discussion after statement of Theorem B). Also, it takes three steps to characterize Lebesgue density points instead of the seven equivalences in §4 of [13]. We think that using the weak integrability notion defined in [7], it should be possible to push our argument to the case of center bundles of higher dimensions.

QUESTION 2.1. *Is it possible to use the techniques here and avoid the fake foliations in case the bunching conditions in [13] hold and E^c is weakly integrable, that is, there are center leaves everywhere tangent to E^c at every point?*

Indeed, Proposition 3.4. of [7] says that when the center bundle is one dimensional, then it is weakly integrable, and this is what allows us to avoid the use of the fake foliations.

Let us consider a diffeomorphism f having the *essential accessibility* property, that is, such that each measurable s- and u-saturated set is of full or null measure. In order to prove that f is *ergodic* (each *invariant* set is of full or null measure) it suffices to show, due to Birkhoff's Ergodic Theorem, that

$$\phi_\pm(x) = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \phi(f^{\pm k}(x)) = \int_M \phi\, \mathrm{dm} \qquad m \text{ a.e. } x$$

for all C^0 observables $\phi : M \to \mathbb{R}$. It is not hard to see that, for each $c \in \mathbb{R}$, the set $S(c) = \phi_+^{-1}(c,\infty)$ is s-saturated, and the set $U(c) = \phi_-^{-1}(c,\infty)$ is u-saturated. Since $m(S(c)\triangle U(c)) = 0$ due to Birkhoff's Theorem, we have that the set $S(c)\cap U(c)$ differs in a set of null measure from an s- saturated set, and also from a u-saturated set. In general, we shall say that a measurable set X is *essentially σ-saturated* if there exists a measurable σ-saturated set X_σ (an *essential σ-saturate of* X) such that $m(X\triangle X_\sigma) = 0$. In short, $S(c)\cap U(c)$ is essentially s- and essentially u-saturated (with essential s-and u- saturates $S(c)$ and $U(c)$, respectively).

Pugh and Shub's adaptation of the usual Hopf's argument goes on by showing that the set of Lebesgue density points of any essentially s- and essentially u-saturated set X is in fact s- and u-saturated, whence the essential accessibility property directly implies ergodicity. When the strong invariant bundles are smooth this follows directly from the differentiability of holonomies. However, in the general case the holonomy maps *are not* differentiable and this problem is overcome using other notions of density points called julienne density points introduced by Pugh and Shub.

Proposition B.1. *The Lebesgue density points of any essentially s- and essentially u-saturated set X form an s- and u-saturated set.*

That is, Lebesgue density points of essentially $s-$ and essentially $u-$saturated sets flow through stable and unstable leaves. As we have said before, Pugh and Shub had suggested in [29] that certain shapes called *juliennes* would be more natural, rather than merely Riemannian balls, in order to treat preservation of density points. Here we follow this line and use certain solid juliennes instead of balls.

Of course, these new neighborhood bases will define different sets of density points. We will consider the following generalization of Lebesgue density points:

Let us say that a point x is a *C_n-density point* of a set X if $\{C_n(x)\}_n$ is a local neighborhood basis of x, and

$$\lim_{n\to\infty} \frac{m(X\cap C_n(x))}{m(C_n(x))} = 1.$$

In particular, the *Lebesgue density points* will be the $\{B_{r^n}(x)\}_{n\geq 1}$-density points, where $B_{r^n}(x)$ is the Riemannian ball centered at x with radius r^n, $r\in(0,1)$. The choice of r is irrelevant, since x is a B_{r^n}-density point of X if and only if

$$\lim_{\varepsilon\to 0} \frac{m(X\cap B_\varepsilon(x))}{m(B_\varepsilon(x))} = 1.$$

A *cu-julienne* $J_n^{cu}(x)$ of x is a dynamically defined local unstable saturation of a center curve, its radius depending on x and n, and going to 0 subject to certain rates related to contraction rates in the bundles (see precise definitions in §4.1, formulas (4.6)). We shall define a ***solid julienne*** $J_n^{scu}(x)$ of x as a local stable saturation of some cu-julienne (precise definitions in §4.3). Let us point out that the definition of $J_n^{scu}(x)$ is not symmetric under exchange of u and s and dual juliennes $J_n^{usc}(x)$ will also be used. The family $\{J_n^{scu}(x)\}_{n\geq 1}$ is a measurable neighborhood basis of x. For this family we obtain

PROPOSITION B.2. *The set of J_n^{scu}-density points of an essentially s-saturated set X is s-saturated.*

By changing the neighborhood basis, we have solved the problem of preserving density points, that is we have established Proposition B.1 but for julienne density points. However, we need to know now what the relationship is between the julienne density points, and Lebesgue density points. Given a family $\mathcal{M}$ of measurable sets, let us say that two systems $\{C_n\}_n$ and $\{E_n\}_n$ are *Vitali equivalent* over $\mathcal{M}$, if the set of C_n-density points of X equals (as sets, not only a.e.) the set of E_n-density points of X for all $X \in \mathcal{M}$. The argument is completed by showing that

PROPOSITION B.3. *The family $\{J_n^{scu}(x)\}$ is Vitali equivalent to Lebesgue over essentially u-saturated sets.*

Hence, over essentially s- and u-saturated sets, the set of Lebesgue density point is s- saturated. A symmetric argument shows it is also u-saturated.

This ends the proof of Proposition B.1 and, actually, it shows essential accessibility implies ergodicity. To show that, in fact, it implies Kolmogorov property, [25] states that it suffices to see that the Pinsker algebra (the largest subalgebra for which the entropy is zero) is trivial. But after [8], sets in the Pinsker algebra are essentially s- and essentially u-saturated, which proves Theorem B.

3. ACCESSIBILITY IS C^r OPEN AND DENSE

Let $AC(x)$ denote the accessibility class of the point x. We will show that the set

$$\mathscr{D} = \{f \in \mathrm{PH}^r_m(M) : AC(x) \text{ is open for all } x \in \mathrm{Per}(f)\}$$

is C^r generic, where $\mathrm{Per}(f)$ denotes the set of periodic points of f. Afterwards, as stated in Proposition A.1, it will be shown that $\mathscr{D}$ may be decomposed into a disjoint union

$$\mathscr{D} = \mathscr{A} \cup \mathscr{B} \tag{3.1}$$

where $\mathscr{A}$ consists of diffeomorphisms with the accessibility property and $\mathscr{B}$ consists of diffeomorphisms without periodic points and satisfying that the distribution $E^s \oplus E^u$ is integrable. Moreover, $\mathscr{B}$ will be shown to be meager. This will prove Proposition A.2 and, in fact, Theorem A.

We shall set for a given subset $X \subset M$,

$$\mathcal{W}^\sigma_{loc}(X) = \bigcup_{x \in X} \mathcal{W}^\sigma_{loc}(x) \qquad \text{with} \quad \sigma = s, u.$$

3.1. **A lamination in the complement of open accessibility classes.** Fix $f \in \mathrm{PH}^r_m(M)$, and let $U(f)$ be the set of points whose accessibility classes are open, and $\Gamma(f) = M \setminus U(f)$ be the complement of $U(f)$. We say that a partition $\mathcal{L}$ of a set $N \subset M$ by disjoint pathwise-connected subsets is a *lamination* if for every $x \in N$ there is a neighborhood of x, $U_x \subset N$, a set $T_x \subset N$ containing x, called a *transversal*, and a homeomorphism $\phi_x : T_x \times D \to U_x$ where $D \subset \mathbb{R}^k$ is a neighborhood of 0, such that for every $z \in T_x$,

$$\phi_x(z, 0) = z, \quad \phi_x(z \times D) \subset \mathcal{L}(z)$$

and $\phi_x(z \times D)$ is the connected component of $\mathcal{L}(z) \cap U_x$ containing z, here $\mathcal{L}(y)$ denotes the element of the partition containing y. We say that each element of the partition is a lamina or a leaf. Observe that, even though we do not require any smoothness of the lamination, in our case the leaves will be C^1 manifolds. But we shall not need this property here.

Proposition A.3. *$\Gamma(f)$ is a compact, invariant set laminated by the accessibility classes.*

Let us begin the proof with the following general proposition valid for any center dimension.

Proposition A.4. *For a given point $x \in M$ the following statements are equivalent*

1. *$AC(x)$ is open.*
2. *$AC(x)$ has non empty interior.*
3. *$AC(x) \cap W^c_{loc}(x)$ has nonempty interior for any choice of $W^c_{loc}(x)$.*

The idea of the proof of this proposition is already in [32, Section 5].

Proof. We shall prove that 2) $\Rightarrow$ 1) $\Rightarrow$ 3) $\Rightarrow$ 2).

To begin with the proof, let us see that 2) $\Rightarrow$ 1). Let y be in the interior of $AC(x)$, take $z \in AC(x)$ and let us see that z is in the interior of $AC(x)$. Take an su-path joining z and y. Let $z = x_0, x_1, \dots, x_{n-1}, x_n = y$ be points in the su-path such that x_i and x_{i+1} are in the same σ-leaf (for σ either s or u). Take $U \subset AC(x)$ an open neighborhood of y. Let us define inductively $U_n = U$ and $U_i = \mathcal{W}^\sigma(U_{i+1})$, $i = 1, \dots, n$, where σ is s or u depending on whether the x_i is in the s or u-leaf of x_{i+1} respectively. Observe that $x_i \in U_i$ for every i and that

U_i is open for every i since the strong manifolds $\mathcal{W}^\sigma$ form C^0 foliations. Hence, since $U_i \subset AC(x)$ for every i we get that $z \in U_0$ and $U_0 \subset AC(x)$ and hence z is in the interior of $AC(x)$ (see Figure 1).

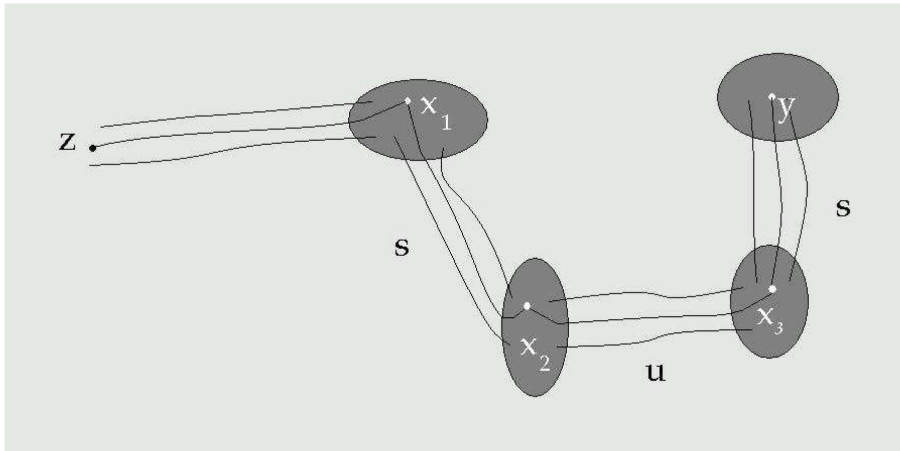

Figure 1. An su path from z to y

1) $\Rightarrow$ 3) follows from definition of relative topology since $x \in AC(x)$ and $AC(x)$ is open.

So let us prove that 3) $\Rightarrow$ 2). Take $W^c_{loc}(x)$ and assume that there is an open set $V \subset W^c_{loc}(x) \cap AC(x)$. To prove that $AC(x)$ has nonempty interior, let us define, for

$$W^{sc}_{loc}(x) = \mathcal{W}^s_{loc}(W^c_{loc}(x)) \text{ and } W^{usc}_{loc}(x) = \mathcal{W}^u_{loc}(W^{sc}_{loc}(x))$$

the map

$$p_{us} : W^{usc}_{loc}(x) \to W^c_{loc}(x)$$

that is obtained by first projecting along $\mathcal{W}^u$ and then along $\mathcal{W}^s$ (see Figure 2). The map is continuous because the strong foliations are C^0. Also because $\mathcal{W}^\sigma_{loc}(x)$ depends continuously on x, it follows that $W^{usc}_{loc}(x)$ is an open set. So we have that $p^{-1}_{us}(V)$ is open and $p^{-1}_{us}(V) \subset AC(x)$. But $p^{-1}_{us}(V)$ is clearly inside $AC(x)$, hence $AC(x)$ has nonempty interior. $\square$

Proof of Proposition A.3. Let $AC_x(y)$ denote the connected component of $AC(y) \cap W^{usc}_{loc}(x)$ containing y. We shall also make use of the following:

Theorem 2. [18] *If $f \in \mathrm{PH}^r_m(M)$ then there are continuous functions $\gamma^\sigma : M \to Emb^r(D^\sigma, M)$, $\sigma = s, u$, where $Emb^r(D^\sigma, M)$ is the set of C^r embeddings from the disk of dimension σ into M, such that $\mathcal{W}^\sigma_{loc}(z) = \gamma^\sigma(z)(D^\sigma)$ for $\sigma = s, u$.*

Let us see that the partition $AC(x)$ forms a lamination of $\Gamma(f)$. Given a point $x \in \Gamma(f)$ let us fix a center curve through x, $W^c_{loc}(x)$, and take the neighborhood

$\widehat{U}_x = W^{usc}_{loc}(x)$. Take the transversal $\widehat{T}_x = W^c_{loc}(x)$, and the disk $D = D^s \times D^u$ where D^σ is as in Theorem 2, $\sigma = s, u$. Define the homeomorphism $\phi : \widehat{T}_x \times D \to \widehat{U}_x$ by

$$\phi(z,t) = \gamma^u \left(\gamma^s(z)(t^s)\right)(t^u),$$

where $t = (t^s, t^u)$. ϕ is continuous since γ^σ are continuous. ϕ is clearly onto. Let us see that ϕ is 1 to 1. Assume that $\phi(z,t_1) = \phi(z',t_2) = y$, where $t_i = (t^s_i, t^u_i)$, $i = 1,2$. Then we have that $z = z'$ since $p_{us}(\phi(z,t)) = z$ for every z and t. By transversality we have that

$$\gamma^s(z)(t^s_i) = \mathcal{W}^u_{loc}(y) \cap W^{sc}_{loc}(c)$$

for $i = 1,2$ and hence $\gamma^s(z)(t^s_1) = \gamma^s(z)(t^s_2)$ and hence $t^s_1 = t^s_2$ because $\gamma^s(z)$ is injective. So we get that

$$\gamma^u(\gamma^s(z)(t^s_1))(t^u_1) = \gamma^u(\gamma^s(z)(t^s_1))(t^u_2)$$

and this implies again that $t^u_1 = t^u_2$ by injectivity of $\gamma^u(\gamma^s(z)(t^s_1))$. Hence ϕ is a homeomorphism by the invariance of domain theorem. Clearly $\phi(z,0) = z$. So, if we set $U_x = \widehat{U}_x \cap \Gamma(f)$ the neighborhood of x in $\Gamma(f)$, and $T_x = \widehat{T}_x \cap \Gamma(f)$ the transversal, we get that the restriction of $\phi : T_x \times D \to U_x$ is a homeomorphism. Observe that

$$\phi(z \times D) = \mathcal{W}^u_{loc}(\mathcal{W}^s_{loc}(z)) = p^{-1}_{us}(z).$$

Let us see that $\phi(z \times D) = AC_x(z)$. This is done in the following lemma.

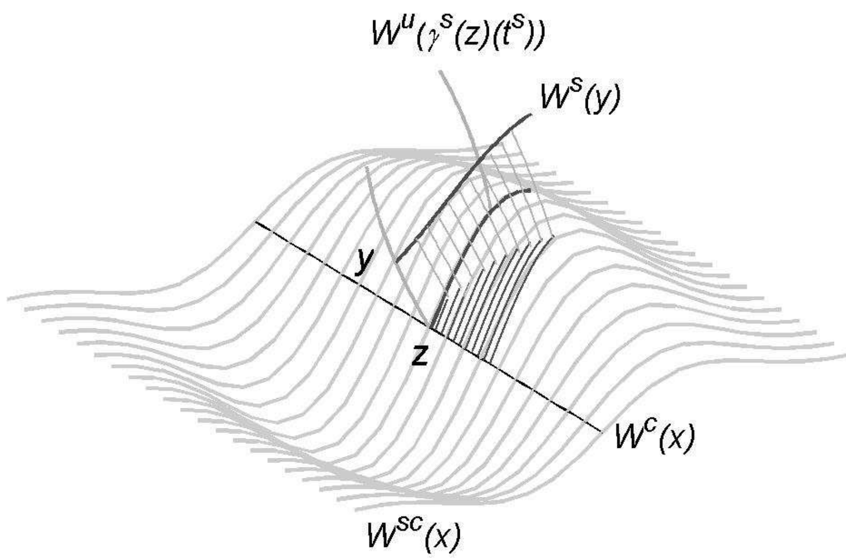

FIGURE 2. A point in $U(f)$ (open accessibility class)

LEMMA A.4.1. *If $z \in \Gamma(f) \cap W^c_{loc}(x)$ then $p^{-1}_{us}(z) = AC_x(z)$.*

Proof. Since $AC_x(z)$ is connected and p_{us} is continuous we have that $p_{us}(AC_x(z))$ is a connected subset of $W^c_{loc}(x)$, see figure 2. Thus if $p_{us}(AC_x(z))$ contains more than one point, it would have non empty interior since $W^c_{loc}(x)$ is one dimensional. Hence, by Proposition A.4 $AC(z)$ would be open, contradicting that $z \in \Gamma(f)$. Since z is clearly in $AC_x(z)$ and $p_{us}(z) = z$ we have that $p_{us}(AC_x(z)) = z$ and hence $AC_x(z) \subset p_{us}^{-1}(z)$. The other inclusion follows since $p_{us}^{-1}(z) = \mathcal{W}^u_{loc}(\mathcal{W}^s_{loc}(z))$ is connected and $p_{us}^{-1}(z) \subset AC(z) \cap W^{usc}_{loc}(x)$. □

So we get that $\Gamma(f)$ is laminated by the accessibility classes and this ends the proof of Proposition A.3. □

Let us observe the following consequence of our proof.

REMARK 3.1. *There is an $\varepsilon > 0$ such that if $x \in \Gamma(f)$, $y \in \mathcal{W}^u_{loc}(x)$ and $z \in \mathcal{W}^s_{loc}(x)$ are ε-close to x then*

$$\mathcal{W}^s_{loc}(y) \cap \mathcal{W}^u_{loc}(z) \neq \emptyset,$$

see Figure 3.

Proof. Let us first see that given two points, a and b such that $a \in \Gamma(f)$ and $b \in \mathcal{W}^u_\varepsilon(a)$, we have that $\mathcal{W}^s_\varepsilon(b) \subset \mathcal{W}^u_{loc}(\mathcal{W}^s_{loc}(a))$. In fact, for some $\varepsilon > 0$ small, $\mathcal{W}^s_\varepsilon(b) \subset W^{usc}_{loc}(a)$, hence, since $b \in \mathcal{W}^u_{loc}(a)$, we get that $\mathcal{W}^s_\varepsilon(b) \subset AC_a(a)$ and so $p_{us}(\mathcal{W}^s_\varepsilon(b)) = a$, that is, $\mathcal{W}^s_\varepsilon(b) \subset \mathcal{W}^u_{loc}(\mathcal{W}^s_{loc}(a))$.

So, taking $a = y$ and $b = x$ we have proved that $\mathcal{W}^s_\varepsilon(x) \subset \mathcal{W}^u_{loc}(\mathcal{W}^s_{loc}(y))$. Hence, if $z \in \mathcal{W}^s_\varepsilon(x)$, then $z \in \mathcal{W}^u_{loc}(r)$ for some $r \in \mathcal{W}^s_{loc}(y)$. But then, $r \in \mathcal{W}^u_{loc}(z)$ and $r \in \mathcal{W}^s_{loc}(y)$, and hence $r \in \mathcal{W}^s_{loc}(y) \cap \mathcal{W}^u_{loc}(z) \neq \emptyset$. □

3.2. **Keepaway Lemma.** Let f be a diffeomorphism preserving a foliation $\mathcal{W}$ tangent to a continuous sub-bundle $E \subset TM$. Call $\mathcal{W}(x)$ the leaf of $\mathcal{W}$ through

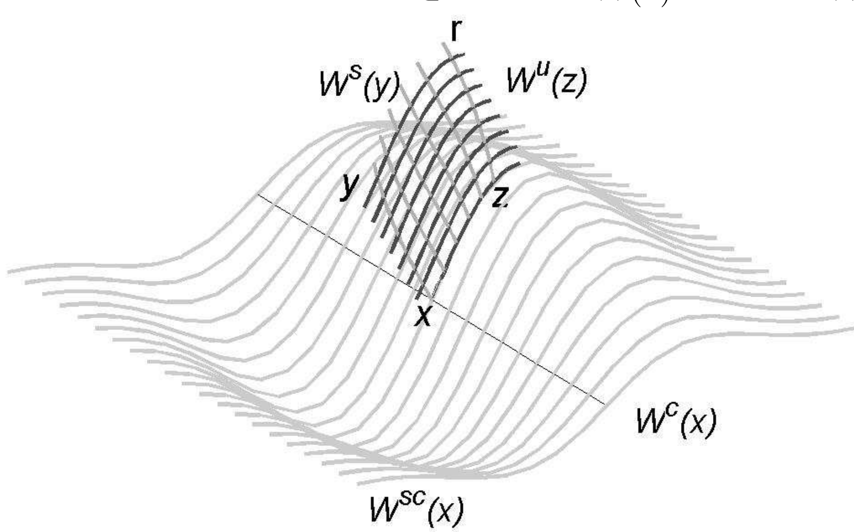

FIGURE 3. A point in $\Gamma(f)$

x and $\mathcal{W}_\varepsilon(x)$ the set of points that are reached from x by a curve contained in $\mathcal{W}(x)$ of length less than ε. Given a (small) disk V transverse to $\mathcal{W}$ whose dimension equals the codimension of E, define $B_\varepsilon(V) = \cup\{\mathcal{W}_\varepsilon(y);\ y \in V\}$; also define $C_\varepsilon(V) = B_{5\varepsilon}(V) \setminus B_\varepsilon(V)$.

The following lemma was already proved by R. Mañé in [22, Lemma 5.2.] when the dimension of E is 1. His proof generalizes to higher dimensions with some changes and here we present this generalization.

Lemma A.4.2 (Keepaway Lemma). *Let us assume that $||Tf^{-1}|_E|| < \mu < 1$. Let N be such that $\mu^{-N} > 5$. Given V a small disk transverse to $\mathcal{W}$ and $\varepsilon > 0$, if*

$$f^n(C_\varepsilon(V)) \cap B_\varepsilon(V) = \emptyset \qquad \forall n = 1, \dots, N$$

then given $y \in V$ there is $z \in \mathcal{W}_{5\varepsilon}(y) \setminus \mathcal{W}_\varepsilon(y)$ such that $f^n(z) \notin B_\varepsilon(V)$ for all $n \geq 0$.

Proof. Let $y \in V$ and $w \in \mathcal{W}_{5\varepsilon}(y)$ be such that $\mathcal{W}_\varepsilon(w) \subset C_\varepsilon(V)$. Set $D_0 = \overline{\mathcal{W}_\varepsilon(w)}$. We shall construct, by induction, a sequence of closed disks D_n such that $f^{-1}(D_n) \subset D_{n-1}\ \forall n > 0$ and $D_n \cap B_\varepsilon(V) = \emptyset$. Thus z will be any point in $\bigcap\{f^{-n}(D_n);\ n \in \mathbb{N}\}$ (in fact in our construction this intersection will consist of a unique point).

For the following construction, observe that for any δ and $x \in M$ we have that $\overline{\mathcal{W}_\delta(f(x))} \subset \mathcal{W}_{\mu^{-1}\delta}(f(x)) \subset f(\mathcal{W}_\delta(x))$. The construction is as follows:

(1) If $n < N$ put $D_n = f^n(D_0)$.

(2) For the N^{th} iterate, observe that still $f^N(D_0) \cap B_\varepsilon(V) = \emptyset$ but $f^N(D_0)$ contains a round ball around $f^N(w)$ of radius 5ε, that is,

$$\overline{\mathcal{W}_{5\varepsilon}(f^N(w))} \subset f^N(D_0).$$

So we may change from iterates of D_0 to round balls, that is, put $D_n = \overline{\mathcal{W}_{5\varepsilon}(f^n(w))}$ for $n = N, \dots, n_1 - 1$ where n_1 is the first iterate such that $\overline{\mathcal{W}_{5\varepsilon}(f^{n_1}(w))} \cap B_\varepsilon(V) \neq \emptyset$. Observe that $D_n \subset f(D_{n-1})$ and $D_n \cap B_\varepsilon(V) = \emptyset$ for $n = 0, \dots, n_1 - 1$.

(3) For the n_1^{th} iterate, we can not take $\overline{\mathcal{W}_{5\varepsilon}(f^{n_1}(w))}$, since this disk intersects $B_\varepsilon(V)$. So, either the intersection $\overline{\mathcal{W}_\varepsilon(f^{n_1}(w))} \cap B_\varepsilon(V)$ is empty or not. If it is empty then take the point $w_{n_1} = f^{n_1}(w)$ and $D_{n_1} = \overline{\mathcal{W}_\varepsilon(w_{n_1})}$. If it is nonempty, then any point $w_{n_1} \in \mathcal{W}_{5\varepsilon}(f^{n_1}(w))$ with $d(w_{n_1}, f^{n_1}(w)) = 4\varepsilon$ will satisfy $\overline{\mathcal{W}_\varepsilon(w_{n_1})} \subset C_\varepsilon(V)$, see Figure 4, so take this point w_{n_1} and take $D_{n_1} = \overline{\mathcal{W}_\varepsilon(w_{n_1})}$. Observe that in either case

$$f(D_{n_1 - 1}) \supset \mathcal{W}_{5\varepsilon}(f^{n_1}(w)) \supset D_{n_1}.$$

(4) Now, to continue the construction, go to step 1, replace D_0 by D_{n_1} and w by w_{n_1}.

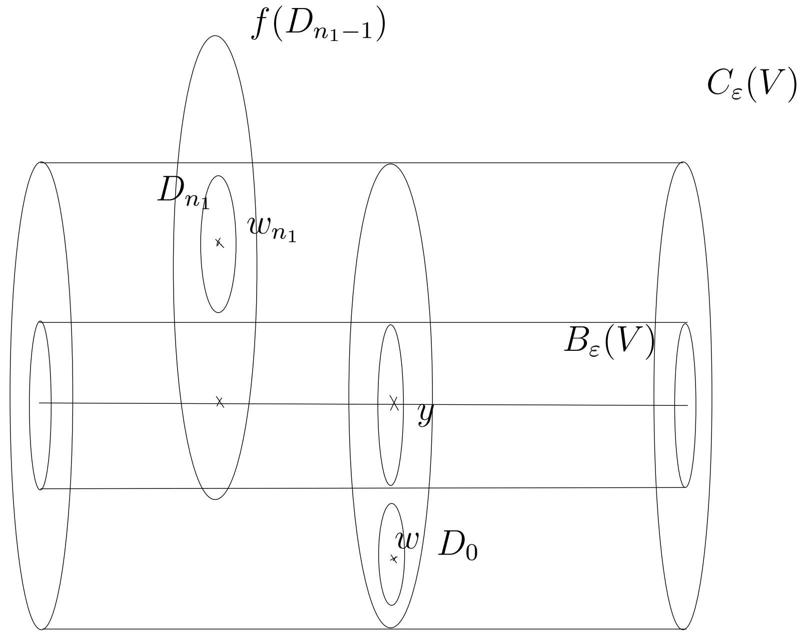

FIGURE 4. Keepaway Lemma

This algorithm gives the desired sequence of disks, and hence the point z, proving the lemma. □

We would like to thank Keith Burns for pointing out some inaccuracies in first versions of the proof of the Keepaway Lemma and the the following Corollary.

We have the following corollary of the Keepaway Lemma that deals with the abundance of nonrecurrent points.

COROLLARY A.1. *Let $f : M \to M$ leave invariant an expanding foliation $\mathcal{W}$.*

(1) *For every $x \in M$ the set of points $\{y \in \mathcal{W}(x) : y \notin \omega(y)\}$ is dense in $\mathcal{W}(x)$, that is, the points that are nonrecurrent in the future are dense in $\mathcal{W}(x)$ for every x.*
(2) *If f is partially hyperbolic then for every $x \in M$ and for every $\varepsilon > 0$ there is a point $y \in \mathcal{W}^s_\varepsilon(\mathcal{W}^u_\varepsilon(x))$ such that $y \notin \omega(y)$ and $y \notin \alpha(y)$, in particular, the nonrecurrent points (for the future and the past) are dense in M and can be found in any accessibility class.*

Proof. Let us prove the first property. Take a point x and le us prove that the points that are not recurrent in the future are dense in $\mathcal{W}(x)$. Take $z \in \mathcal{W}(x)$ and let us approach it inside $\mathcal{W}(x)$ by points nonrecurrent in the future. We may assume z is not periodic since periodic points can always be approached by non-periodic ones inside the same $\mathcal{W}$ leaf. If $z \notin \omega(z)$ then z approaches itself. So, let us assume that $z \in \omega(z)$.

Take ε_0 such that the ε_0 ball around z does not return in the first N iterates, where N is the N of the Keepaway Lemma (this is always possible since z is not periodic). Let us fix ε much smaller than ε_0, something like $\varepsilon_0/100$. Now, since z is forward recurrent, take a very big positive iterate $f^n(z)$ such that the distance between z and $f^n(z)$ is so small that if we take a small transversal V through $f^n(z)$, $C_\varepsilon(V) \subset B_{\varepsilon_0}(z)$. Moreover, we may require also that $B_{\varepsilon/2}(z) \subset B_\varepsilon(f^n(z))$.

Using the Keepaway Lemma, take a point $\bar{y}$ in $W_{5\varepsilon}(f^n(z))$ such that the positive orbit of $\bar{y}$ do not enter $B_\varepsilon(f^n(z))$. Take $y = f^{-n}(\bar{y})$ and observe that y is as close as wanted to z because n is as big as we want, so, in particular, we may assume y is in $B_{\varepsilon/2}(z)$. Finally, y cannot be forward recurrent since if it were recurrent then the forward orbit of $\bar{y}$ will approach y and hence should enter $B_{\varepsilon/2}(z) \subset B_\varepsilon(f^n(z))$.

The second property is an application of the first and of the fact that if a point y does not return in the future to a small neighborhood of y, then points in its stable manifold do not return either. $\square$

Set $\mathscr{I} = \{f \in \mathrm{PH}^r_m M;\ E^s \oplus E^u \text{ is integrable}\}$. Observe that $\mathscr{I}$ is a closed set and $\mathscr{B} \subset \mathscr{I}$ (see definition of $\mathscr{B}$ in page 8).

In the partially hyperbolic setting the Keepaway Lemma A.4.2 has as corollaries that $\mathscr{I}$ has empty interior and that, given a periodic point x, f can be perturbed in such a way that the accessibility class of x for the perturbed diffeomorphism is open. This is shown in the following subsections.

3.3. $\mathscr{D}$ is generic.

After the following property, genericity of $\mathscr{D}$ follows from a Baire type argument like in the proof of Kupka-Smale Theorem:

Lemma A.4.3 (Unweaving Lemma). *For each $x \in \mathrm{Per}(f)$ there exists g C^r-close to f such that $x \in \mathrm{Per}(g)$ and $AC_g(x)$ is open.*

Proof. Assume that $AC_f(x)$ is not open for some periodic point x of period k. Then, as stated in Remark 3.1, for all $y \in \mathcal{W}^u_{loc}(x)$ and all $z \in \mathcal{W}^s_{loc}(x)$ ε-close to x, $\mathcal{W}^s_{loc}(y) \cap \mathcal{W}^u_{loc}(z) \neq \emptyset$.

The idea is to perturb a small neighborhood of x, so that $x \in \mathrm{Per}(g)$ and $\mathcal{W}^s_{g,loc}(y') \cap \mathcal{W}^u_{g,loc}(z') = \emptyset$ for some $y' \in \mathcal{W}^u_{g,loc}(x)$ and $z' \in \mathcal{W}^s_{g,loc}(x)$ close to x. This will obviously prove $AC_g(x)$ is open.

We shall use the Keepaway Lemma A.4.2 to find these points. To find a suitable transversal V let us state the following:

Theorem 3. [18] *Center-Stable Manifold. Given a periodic point of period k whose derivative admits an invariant by $D_{f^j(x)}f^k$ splitting $T_{f^j(x)}M = E^s_{f^j(x)} \oplus E^c_{f^j(x)} \oplus E^u_{f^j(x)}$, for $j = 0, \dots, k-1$, which is partially hyperbolic, there are center-stable manifolds $W^{cs}_{loc}(f^j(x))$ tangent to $E^s_{f^j(x)} \oplus E^c_{f^j(x)}$ at $f^j(x)$ satisfying the following: for every $N > 0$ there is $\delta > 0$ such that if $0 \le j \le N$, then $f^j(W^{cs}_\delta(x)) \subset W^{cs}_{loc}(f^j(x))$ where we denote by $W^{cs}_\delta(f^j(x))$ the center-stable manifold $W^{cs}_{loc}(f^j(x))$ intersected with the ball centered at $f^j(x)$ and radius δ.*

FIGURE 5. Unweaving Lemma: Before and after perturbing around a periodic point x

So we shall take $V = W^{cs}_{\delta}(x)$ where δ is as in the Center-Stable Manifold Theorem for the N of the Keepaway Lemma. Observe that $W^{cs}_{loc}(x)$ is a priori different from $W^{sc}_{loc}(x)$ and hence we know it is tangent to $E^c \oplus E^s$ only at x, but this is enough because we only require V to be transversal to E^u.

Now, if we take ε small in the Keepaway Lemma, it is not hard to see that V satisfies its hypotheses. Hence, we obtain a point $y \in \mathcal{W}^u_{5\varepsilon}(x)$ such that its forward orbit does not intersect $B_\varepsilon(V)$. Analogously, applying the Keepaway Lemma to f^{-1}, we obtain a point $z \in \mathcal{W}^s_{5\varepsilon}(x)$ that does not return for the past to a similar neighborhood of x, say $B_\varepsilon(\hat{V})$.

Now, if we take j big enough, for some $\delta > 0$ small, we have that $f^{-jk}(y)$ and $f^{jk}(z)$ are very close to x and hence $\mathcal{W}^u_\delta(f^{jk}(z)) \cap \mathcal{W}^s_\delta(f^{-jk}(y))$ is not empty by Remark 3.1; let us denote the point of intersection by w, see the left figure in Figure 5. Let $y' = f^{-jk}(y)$ and $z' = f^{jk}(z)$. We may assume that $\delta > 0$ is so small that $\mathcal{W}^s_\delta(y')$, $\mathcal{W}^u_\delta(z')$ and w are contained in $B_\varepsilon := B_\varepsilon(V) \cap B_\varepsilon(\hat{V})$.

Since $y \in \mathcal{W}^u_{5\varepsilon}(x)$ does not return to $B_\varepsilon(V)$ in the future and $z \in \mathcal{W}^s_{5\varepsilon}(x)$ does not return to $B_\varepsilon(\hat{V})$ in the past, we can choose U, a sufficiently small neighborhood of w, in such a way that $f^n(\mathcal{W}^s_\delta(y'))$ and $f^{-n}(\mathcal{W}^u_\delta(z'))$ does not intersect U for all $n \geq 1$. Also we can require U not to intersect $\mathcal{W}^\sigma_\varepsilon(f^n(x))$ for $\sigma = u, s$, (see Figure 6).

We now show that if we perform any perturbation whose support is in U then $\mathcal{W}^s_\varepsilon(x)$, $\mathcal{W}^u_\varepsilon(x)$, $\mathcal{W}^u_\delta(z')$ and $f(\mathcal{W}^s_\delta(y'))$ do not change, i.e. if $g = f \circ h$ for some diffeomorphism h satisfying $h(a) = a$ for every $a \notin U$ then

$$\mathcal{W}^s_\varepsilon(x, g) = \mathcal{W}^s_\varepsilon(x, f), \;\; \mathcal{W}^u_\varepsilon(x, g) = \mathcal{W}^u_\varepsilon(x, f), \;\; \mathcal{W}^u_\delta(z', g) = \mathcal{W}^u_\delta(z', f)$$

and $\mathcal{W}^s_\delta(y', g) = h^{-1}(\mathcal{W}^s_\delta(y', f))$.

In fact, observe that if a set A is such that $f^n(A) \cap U = \emptyset$ for every $n \geq 0$, then $f^n|_A = g^n|_A$ for every $n \geq 1$. In the same way, if $f^{-n}(A) \cap U = \emptyset$ for every $n \geq 1$, then $f^{-n}|_A = g^{-n}|_A$ for every $n \geq 1$. This, plus the characterization of the strong invariant manifolds and the choice of U gives us the first three identities. The

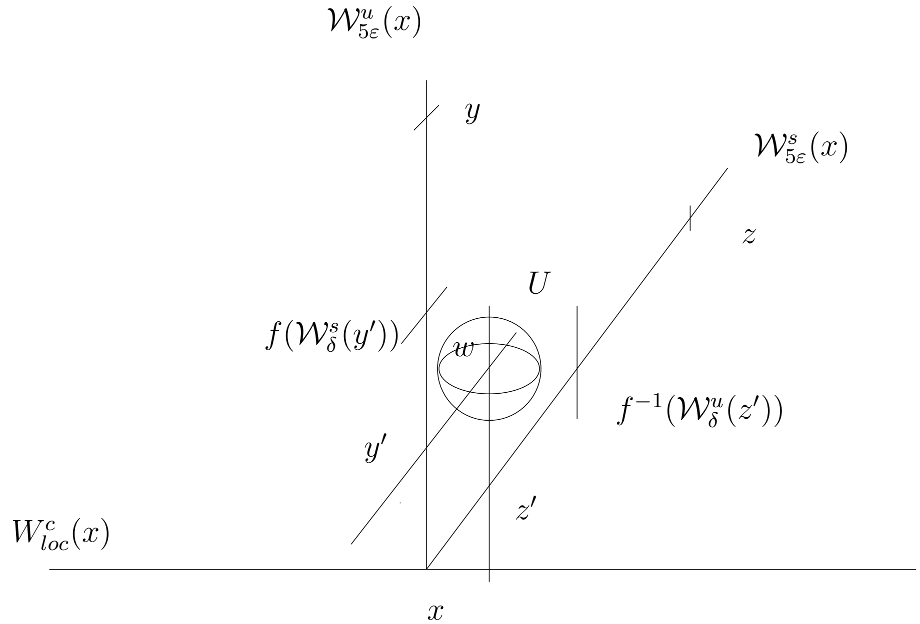

FIGURE 6. Unweaving Lemma: the neighborhood U

fourth one follows essentially in the same manner. In fact, by the characterization of the strong invariant manifolds, to prove the forth identity it is enough to see that $g^n(h^{-1}(\mathcal{W}^s_\delta(y', f))) = f^n(\mathcal{W}^s_\delta(y', f))$ for every $n \geq 1$. This is obvious for $n = 1$ by definition of g, and for $n \geq 2$ because $f^n(f(\mathcal{W}^s_\delta(y', f))) \cap U = \emptyset$ for $n \geq 0$.

So, let us take h to be the time t map of a flow generated by a C^∞ divergence free vector-field X which is 0 outside U and such that $X(w)$ is a unit vector transverse to $E^s \oplus E^u$ at w. h is clearly volume preserving and if t is small enough we have that h is C^∞ close to the identity and hence g is C^∞ close to f.

Moreover, since $X(w)$ is transverse to $E^s \oplus E^u$ at w, if t is small enough, we get that $h(W^u_\delta(z', f)) \cap W^s_\delta(y', f) = \emptyset$. Since $\mathcal{W}^s_\delta(y', g) = h^{-1}(\mathcal{W}^s_\delta(y', f))$ this implies that $W^u_\delta(z', g) \cap W^s_\delta(y', g) = \emptyset$ and this implies, using Remark 3.1, that the periodic point x cannot be in $\Gamma(g)$ and hence that its accessibility class $AC_g(x)$ is open. $\square$

Using a Baire type argument like the one in the proof of Kupka-Smale Theorem, we get, using the Unweaving Lemma above, that C^r-generically

$$\mathrm{Per}(f) \subset U(f).$$

This means, the set $\mathscr{D}$ is C^r-generic. The following proposition shows that, in case $\Gamma(f)$ is a proper subset, there are always periodic points in $\Gamma(f)$. This situation is meager.

Proposition A.5. *If $\emptyset \subsetneq \Gamma(f) \subsetneq M$, then $Per(f) \cap \Gamma(f) \neq \emptyset$.*

For the proof of this proposition let us state the following reformulation of Lemma 2.1 in page 5.

Lemma A.5.1. *For each small $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(y, z) < \delta$ and $z \in W^c_\delta(x)$, then $W^c_{loc}(y) \cap \mathcal{W}^s_\varepsilon(\mathcal{W}^u_\varepsilon(x)) \neq \emptyset$, regardless of the choice of center leaves of x and y, see Figure 7.*

Note that in Figure 7 $x \in \Gamma(f)$, but the lemma holds for all $x \in M$.

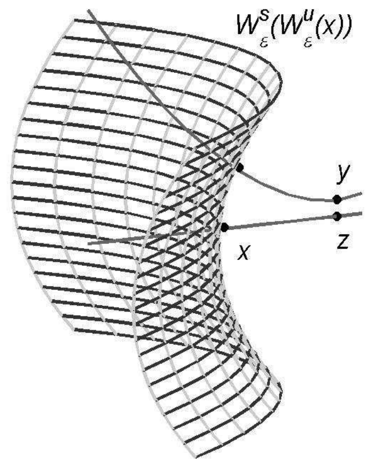

Figure 7. Lemma A.5.1, case $x \in \Gamma(f)$

Proof of Proposition A.5. Let us prove there is a periodic point in the boundary $\partial\Gamma(f)$ of $\Gamma(f)$. Observe that $\partial\Gamma(f)$ is a compact, f-invariant, su-saturated set. We will assume M and E^c are orientable. Indeed, by taking a double covering if necessary, we can assume M is orientable. If E^c is not orientable, we take again a double covering $\tilde{M}$ of M in such a way that $\tilde{E}^c$, the lift of E^c, is orientable. Let $\tilde{f}$ be a lift of f to $\tilde{M}$, then $\tilde{f}^2$ is partially hyperbolic, $\tilde{E}^c$ is its center bundle and $\tilde{f}^2$ preserves the orientation of $\tilde{E}^c$. Any point $x \in \Gamma(f)$ lifts to a point $\tilde{x} \in \tilde{\Gamma}(\tilde{f}^2) \subset \tilde{M}$. The set $\tilde{\Gamma}(\tilde{f}^2)$ is locally diffeomorphic to $\Gamma(f)$, and is $\tilde{f}^2$ invariant. So a periodic point for $\tilde{f}^2$ in $\tilde{\Gamma}(\tilde{f}^2)$ will project to a periodic point for f in $\Gamma(f)$. So we shall assume that M and E^c are orientable, and that f preserves orientation of E^c.

Given any center curve $W^c_{loc}(x)$, we shall identify it with an interval in the line in such a way that the orientation of E^c coincides with the standard orientation in the line. Take a point $x \in \partial\Gamma(f)$ such that there is an open interval $I = (a_x, c_x) \subset W^c_{loc}(x) \setminus \Gamma(f)$ with $a_x = x$, and $c_x \notin \Gamma(f)$. Take $\varepsilon > 0$ so small that $V = \mathcal{W}^s_\varepsilon(\mathcal{W}^u_\varepsilon(I))$ verifies

$$\overline{V} \cap \Gamma(f) = \mathcal{W}^s_\varepsilon(\mathcal{W}^u_\varepsilon(x)) \subset \partial\Gamma(f) \tag{3.2}$$

In what follows, we shall reduce the interval I to an interval of the form (a_x, b_x), with $b_x < c_x$. However, Equality (3.2) holds for any such interval. Note that $f^k(V) \cap \Gamma(f) = \emptyset$ for all $k \in \mathbb{Z}$. Let $\delta > 0$ be as in Lemma A.5.1, and consider the set $U = V \cap B_\delta(x)$. We lose no generality in assuming $I \subset U$. Given $y \in U$ and a choice of $W^c_{loc}(y)$, let us denote by (a_y, b_y) the connected component of the set $W^c_{loc}(y) \cap V$ containing y. Lemma A.5.1 implies that $W^c_{loc}(y) \cap \mathcal{W}^s_\varepsilon(\mathcal{W}^u_\varepsilon(z)) \neq \emptyset$ for all $z \in I$. Since a_y is the left end point of the interval (a_y, b_y) with respect to the orientation of E^c then $a_y \in \mathcal{W}^s_\varepsilon(W^u_\varepsilon(x))$. Note that $(a_y, b_y) \cap \Gamma(f) = \emptyset$.

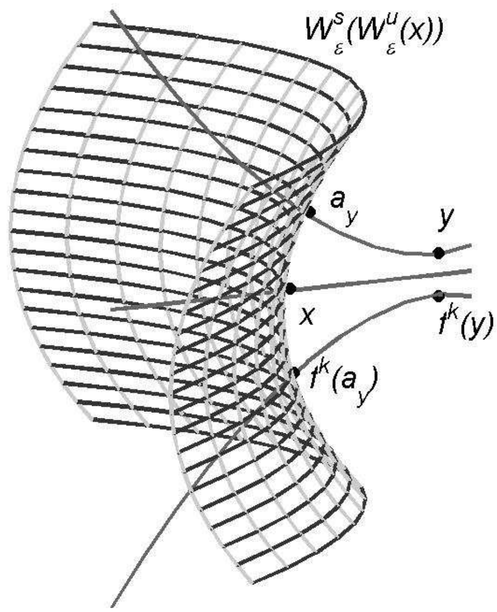

FIGURE 8. Finding periodic orbits - Proposition A.5

Since the non-wandering set of f is M, there exists $y \in U$ such that $f^k(y) \in U$ for some $k > 0$. Now, $f^k(y) \in (f^k(a_y), f^k(b_y)) = f^k(a_y, b_y)$, and $f^k(a_y, b_y) \cap \Gamma(f) = \emptyset$. So, $(f^k(a_y), f^k(y)) \subset (a_{f^k(y)}, b_{f^k(y)})$, whence $f^k(a_y) \in \overline{V}$. On the other hand, $a_y \in \Gamma(f)$, so $f^k(a_y) \in \Gamma(f)$. Therefore, (3.2) implies that $f^k(a_y) \in \mathcal{W}^s_\varepsilon(\mathcal{W}^u_\varepsilon(x))$. In this way, we have shown that there is $a_y \in M$ such that both a_y and $f^k(a_y)$ are in $\mathcal{W}^s_\varepsilon(\mathcal{W}^u_\varepsilon(x))$.

Finally, the proof follows from the following version of the Anosov Closing Lemma (see for instance Lemma 3.8, page 76 in [5] for a proof):

Lemma A.5.2. **Anosov's Closing Lemma** *There is $\varepsilon_0 > 0$ such that if $x \in \Gamma(f)$ satisfies $f^k(\mathcal{W}^s_{\varepsilon_0}(\mathcal{W}^u_{\varepsilon_0}(x))) \cap \mathcal{W}^s_{\varepsilon_0}(\mathcal{W}^u_{\varepsilon_0}(x)) \neq \emptyset$ for some $k > 0$, then there is a periodic point in $\mathcal{W}^s_{\varepsilon_0}(\mathcal{W}^u_{\varepsilon_0}(x))$.*

□

After Proposition A.5, we have the following possibilities for $f \in \mathscr{D}$:

(1) $\Gamma(f) = \emptyset$, that is, f has the accessibility property, i.e. $f \in \mathscr{A}$
(2) $\Gamma(f) = M$ with $\mathrm{Per}(f) = \emptyset$, i.e. $f \in \mathscr{B}$.

The situation $\emptyset \subsetneq \Gamma(f) \subsetneq M$ cannot happen for $f \in \mathscr{D}$, since it implies there is a periodic point in $\Gamma(f)$. This proves Proposition A.1.

3.4. **Proposition A.2.** Recall that $\mathscr{I} = \{f \in \mathrm{PH}^r_m(M);\ E^s \oplus E^u \text{ is integrable}\}$ is a closed set and $\mathscr{B} \subset \mathscr{I}$, next proposition implies Proposition A.2.

Proposition A.6. *$\mathscr{I}$ has empty interior.*

Proof. If there is a periodic point, then we apply the Unweaving Lemma A.4.3 and we get the proposition. So let us assume the set of periodic points is empty.

The proof is similar to that of the Unweaving Lemma A.4.3, but the choice of the points is done in the opposite way. In fact, we first find the point w, which is the point where the perturbation will be realized and then we find the points x, y and z.

Using Corollary A.1 we can find a nonrecurrent point w. Let us then assume that ε is such that $f^n(w) \notin B_\varepsilon(w)$ for every $n \neq 0$. Since w is nonrecurrent and the N in the Keepaway Lemma is fixed, we can find, for some $\varepsilon' < \varepsilon$, points $y \in \mathcal{W}^s_\varepsilon(w)$ and $z \in \mathcal{W}^u_\varepsilon(w)$, $y, z \neq w$, such that $f^{-n}(y) \notin B_{\varepsilon'}(w)$ and $f^n(z) \notin B_{\varepsilon'}(w)$ for $n \geq 1$. Let us take $x = \mathcal{W}^u_{loc}(y) \cap \mathcal{W}^s_{loc}(z)$.

Now, as in the Unweaving Lemma, we can find a small neighborhood $U \subset B_\varepsilon(w)$ of w such that $f^n(\mathcal{W}^s_\varepsilon(y))$ and $f^{-n}(\mathcal{W}^u_\varepsilon(z))$ do not cut U for all $n \geq 1$. Also we can require U not to intersect $f^n(\mathcal{W}^s_{loc}(x))$ and $f^{-n}(\mathcal{W}^u_{loc}(x))$ for all $n \geq 0$. Indeed, let us see how to get that $f^n(\mathcal{W}^s_\varepsilon(y))$ does not cut U for all $n \geq 1$ and that $f^{-n}(\mathcal{W}^u_{loc}(x))$ does not cut U for all $n \geq 0$, the others follow the same idea.

Taking U small enough, we have three possibilities: we have what we want, $f^n(\mathcal{W}^s_\varepsilon(y))$ cuts U infinitely many times, or $w \in f^n(\mathcal{W}^s_\varepsilon(y))$ for some $n > 0$. This last possibility can not occur since in this case we will get that $w \in f^n(\mathcal{W}^s_\varepsilon(y)) \subset f^n(\mathcal{W}^s(w))$ and hence there will be a periodic point. The second possibility can not occur either, because in this case we will get that $\omega(y) \cap U \neq \emptyset$, but $\omega(y) = \omega(w)$ and hence this contradicts that w is nonrecurrent.

Taking U even smaller if necessary we will have what we want or that $f^{-n}(\mathcal{W}^u_{loc}(x))$, $n \geq 0$, cuts U infinitely many times or that for some $n \geq 0$, $w \in f^{-n}(\mathcal{W}^u_{loc}(x))$. In this last case, by the choice of y and z we have that $n \neq 0$, so we will have that $w \in f^{-n}(\mathcal{W}^u_{loc}(x))$ for some $n > 0$, but this implies the existence of a periodic point by the Anosov closing Lemma A.5.2. If $f^{-n}(\mathcal{W}^u_{loc}(x))$, $n \geq 0$, cuts U infinitely many times then $\alpha(x) \cap U \neq \emptyset$ and since $y \in \mathcal{W}^u_{loc}(x)$, then $\alpha(y) \cap U \neq \emptyset$, contradicting the choice of y.

Once we get that the invariant manifolds do not return, the same perturbation as in the Unweaving Lemma works. $\square$

4. Essential accessibility implies ergodicity

4.1. **Definitions.** Let us consider smooth functions $\nu, \hat{\nu}, \gamma, \hat{\gamma} : M \to \mathbb{R}^+$ satisfying, for all unit vectors $v^i \in E^i$ with $i = s, c, u$ and $x \in M$,

$$\|T_x f v^s\| < \nu(x) < \gamma(x) < \|T_x f v^c\| < \hat{\gamma}(x)^{-1} < \hat{\nu}(x)^{-1} < \|T_x f v^u\|$$

where $\nu, \hat{\nu} < 1$ and $\|.\|$ is an adapted Riemannian metric as at the beginning of Section 2. We may also assume that d and $\nu, \hat{\nu}, \gamma, \hat{\gamma}$ satisfy:

$$\begin{aligned} &d(f(x), f(x')) \leq \nu(x)\, d(x, x') &&\text{for} \quad x' \in \mathcal{W}^s_{loc}(x) \\ &d(f^{-1}(x), f^{-1}(x')) \leq \hat{\nu}(f^{-1}(x))\, d(x, x') &&\text{for} \quad x' \in \mathcal{W}^u_{loc}(x) \end{aligned} \tag{4.3}$$

$$\begin{aligned} &d(f(x), f(x')) \leq \hat{\gamma}(x)^{-1} d(x, x') &&\text{for} \quad x' \in W^c_{loc}(x) \\ &d(f^{-1}(x), f^{-1}(x')) \leq \gamma(f^{-1}(x))^{-1} d(x, x') &&\text{for} \quad x' \in W^c_{loc}(x) \end{aligned} \tag{4.4}$$

Remark 4.1. *Inequalities (4.3) and (4.4) do not depend on the choice of the center curve through x.*

Note that $\nu, \hat{\nu} < 1$ and $\gamma\hat{\gamma} < 1$ for any partially hyperbolic diffeomorphism. Moreover, since $\dim E^c = 1$, $\gamma\hat{\gamma}$ may be chosen so close to 1 that $\nu < \gamma\hat{\gamma}$ and $\hat{\nu} < \gamma\hat{\gamma}$. This is the center bunching condition. We can do so since f acts on E^c conformally. If E^c is higher dimensional then this is no longer the case, f may act far from conformally.

Let us introduce a smooth function $\sigma : M \to \mathbb{R}$ satisfying

$$\frac{\nu(x)}{\gamma(x)} < \sigma(x) < \min(1, \hat{\gamma}(x)). \tag{4.5}$$

Consider, for $\alpha = \nu, \hat{\nu}, \gamma, \hat{\gamma}, \sigma$ and $n \geq 0$ the multiplicative cocycles:

$$\alpha_n(x) := \prod_{i=0}^{n-1} \alpha(f^i(x)) \qquad \alpha_{-n}(x) := \alpha_n(f^{-n}(x))^{-1}$$

For each $W^c_{loc}(x)$, define the set

$$B^c_n(x) = W^c_{\sigma_n(x)}(x)$$

and consider also:

$$J_n^u(x) = f^{-n}(\mathcal{W}^u_{\nu_n(x)}(f^n(x))) \qquad \text{and} \qquad J_n^{cu}(x) = \bigcup_{y\in B_n^c(x)} J_n^u(y) \tag{4.6}$$

The sets $J_n^{cu}(x)$ will be called center-unstable juliennes of x or *cu-juliennes*.

We now state two lemmas which will be useful in what follows.

Lemma B.6.1. *For any Hölder continuous $\alpha : M \to \mathbb{R}^+$, there is a fixed constant $C > 1$ such that for all $n \geq 0$, if $y \in \mathcal{W}^s_{loc}(B_n^c(x)) \cup J_n^u(x)$, then*

$$\frac{1}{C} \leq \frac{\alpha_n(x)}{\alpha_n(y)} \leq C$$

Proof. See for instance [12, Proposition 1.6] □

The following is proved in [14, Theorem 0.2.]:

Proposition B.7. *If $f : M \to M$ is $C^{1+\alpha}$ partially hyperbolic with some center bunching condition (trivially satisfied for one-dimensional center bundle) then for any point x admitting a center-stable manifold $W^{sc}_{loc}(x)$ everywhere tangent to $E^s \oplus E^c$ (manifold that always exists when $\dim E^c = 1$, see discussion before Remark 2.1) the stable foliation restricted to $W^{sc}_{loc}(x)$ is C^1 with uniform bounds (here uniform means that do not depend on x).*

In [26, Theorem 2.1.] the reader may find the case when the diffeomorphism is C^2 and the center dimension is 1.

4.2. **Controlling stable holonomy.** In this section we will prove that the deformation suffered by the *cu*-juliennes under the stable holonomy can be controlled in the following sense:

Proposition B.8. *There exists $k \in \mathbb{Z}^+$ such that, if $x' \in \mathcal{W}^s_{loc}(x)$, then for all choices of $W^c_{loc}(x)$ and $W^c_{loc}(x')$ contained in $W^{sc}_{loc}(x)$, the stable holonomy map from $W^{cu}_{loc}(x)$ to $W^{cu}_{loc}(x')$ satisfies*

$$J^{cu}_{n+k}(x') \subset h^s(J_n^{cu}(x)) \subset J^{cu}_{n-k}(x') \qquad \forall\, n \geq k.$$

The proof splits into two parts. On one hand, we prove that the holonomy does not distort center leaves too much, as is seen in Lemma B.8.1. On the other hand, each unstable fiber on a certain center leaf is transformed, under the stable holonomy, into a curve contained in a larger julienne. This is seen in Lemma B.8.2 and Figure 9.

Lemma B.8.1. *There exists $k \in \mathbb{Z}^+$, not depending on x, such that for all choices $W^c_{loc}(x)$, $W^c_{loc}(x')$ of center curves through x, x' contained in some $W^{sc}_{loc}(x)$, with $x' \in \mathcal{W}^s_{loc}(x)$, the stable holonomy map h^s from $W^c_{loc}(x)$ to $W^c_{loc}(x')$, satisfies*

$$h^s(B_n^c(x)) \subset B^c_{n-k}(x') \qquad \forall n \geq k$$

Proof. Recall that by Proposition B.7 the stable holonomy between center manifolds is C^1. Let L be its Lipschitz constant. Let $C > 1$ be as in and Lemma B.6.1. Take $k > 0$ such that $\sigma_{-k}(x) > LC$ for all $x \in M$ (recall that $\sigma < 1$), then

$$h^s(B^c_n(x)) \subset W^c_{L\sigma_n(x)}(x') \subset W^c_{LC\sigma_n(x')}(x') \subset W^c_{\sigma_{n-k}(x')}(x') = B^c_{n-k}(x')$$

and the lemma follows. □

The following lemma is the second part of the proof of Proposition B.8:

LEMMA B.8.2. *There exists $k \in \mathbb{Z}^+$, depending neither on x nor on the choice of the center curves, such that, under the hypotheses of Proposition B.8, the stable holonomy map h^s from $W^{cu}_{loc}(x)$ to $W^{cu}_{loc}(x')$ satisfies*

$$h^s(J^u_n(z)) \subset J^{cu}_{n-k}(x') \qquad \forall n \geq k$$

for all $z \in B^c_n(x)$.

Proof. Consider $x' \in \mathcal{W}^s_{loc}(x)$, and center curves $W^c_{loc}(x)$, $W^c_{loc}(x')$ through x, x' respectively, contained in $W^{cs}_{loc}(x)$. Consider $y \in J^u_n(z)$, with $z \in B^c_n(x)$, and let $y' = h^s(y)$, $z' = h^s(z)$.

Let us find $k > 0$ satisfying:

(1) $y' \in J^u_{n-k}(w') \subset J^{cu}_{n-k}(x')$ with

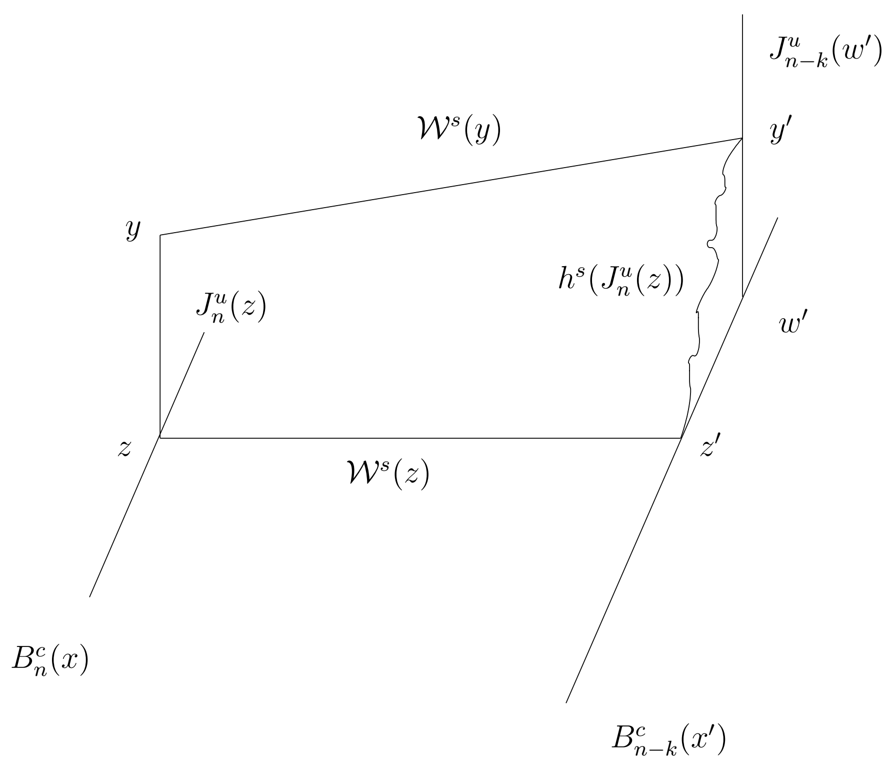

FIGURE 9. Lemma B.8.2

(2) $w' \in B^c_{n-k}(x') \subset W^c_{loc}(x')$.

Since the point $f^n(y)$ is in $\mathcal{W}^u_{\nu_n(z)}(f^n(z))$, we have $d(f^n(y), f^n(z)) \leq \nu_n(z)$. On the other hand, $y' \in \mathcal{W}^s_{loc}(y)$ and $z' \in \mathcal{W}^s_{loc}(z)$ and since f contracts stable manifolds by a factor of ν we have:

$$\begin{aligned} d(f^n(y'), f^n(z')) &\leq d(f^n(y'), f^n(y)) + d(f^n(y), f^n(z)) + d(f^n(z), f^n(z')) \\ &\leq \nu_n(y') + \nu_n(z) + \nu_n(z') \leq K\nu_n(z') \end{aligned}$$

for a fixed constant $K > 0$, not depending on z, z', y, y' (see Lemma B.6.1).

Let $w' \in \mathcal{W}^u_{loc}(y') \cap W^c_{loc}(x')$. Since the angle between E^c and E^u is uniformly bounded from below and since the unstable foliation restricted to W^{uc}_{loc} is uniformly C^1, see Proposition B.7, it follows by projecting along unstable manifolds inside $W^{uc}_{loc}(f^n(x'))$ that there is a constant $C' > 0$ such that

$$d(f^n(y'), f^n(w')) \leq C'\nu_n(y') \qquad \text{and} \qquad d(f^n(w'), f^n(z')) \leq C'\nu_n(z'). \tag{4.7}$$

Hence (1) follows from the first inequality above by taking any $l_0 > 0$ satisfying $\nu_{-l_0}(y) > C'$ for all $y \in M$. Indeed,

$$d(f^{n-l}(y'), f^{n-l}(w')) \leq d(f^n(y'), f^n(w')) \leq C'\nu_n(y') \leq \nu_{n-l}(y')$$

for all $l \geq l_0$. Using Lemma B.6.1 again, one obtains $k > 0$ such that $\nu_{-k}(y) > C$ for all $y \in M$, and so $y' \in J^u_{n-k}(w')$.

From the second inequality in (4.7), and inequalities (4.4) and (4.5) in §4.1 we derive

$$d(w', z') \leq C'\gamma_{-n}(z')\nu_n(z') \leq C'\sigma_n(z') \leq \sigma_{n-l}(z').$$

Now, previous lemma implies $z' \in B^c_{n-l}(x')$ for some sufficiently large $l > 0$, so using Lemma B.6.1 again and taking into account that $z' \in B^c_{n-l}(x')$, we find a (uniform) $k > 0$ so that $d(x', w') \leq \sigma_{n-k}(x')$ for all $n \geq k$. □

4.3. **A characterization of Lebesgue density points.** In this paragraph, we shall see that the following three systems are Vitali equivalent over essentially u-saturated sets:

(1) $Q_n(x) = \bigcup_{w \in J^{sc}_n(x)} \mathcal{W}^u_{\sigma_n(x)}(w)$ where $J^{sc}_n(x) = \bigcup_{y \in B^c_n(x)} \mathcal{W}^s_{\sigma_n(y)}(y)$,
(2) $J^{usc}_n(x) = \bigcup_{y \in J^{sc}_n(x)} J^u_n(y)$,
(3) $J^{scu}_n(x) = \bigcup_{y \in J^{cu}_n(x)} \mathcal{W}^s_{\sigma_n(y)}(y)$.

The first system $Q_n(x)$ consists of "cubic" balls, so it is not difficult to see it is Vitali equivalent to Lebesgue. The second system $J^{usc}_n(x)$ consists of dynamically defined local unstable saturation of local center-stable leafs. Both systems are local unstable saturations of the same center-stable leaf, and in both cases the local unstable fibers are "uniformly" sized, so over essentially u-saturated sets, they have the same density points. This is a consequence of absolute continuity of the unstable foliation. Finally, the systems $J^{usc}_n(x)$ and $J^{scu}_n(x)$ are comparable, in the sense that they are nested, their volumes preserving a controlled ratio. So the three systems are Vitali equivalent over essentially u-saturated sets:

LEMMA B.8.3. *The system $\{Q_n(x)\}_{x\in M}$ is Vitali equivalent to Lebesgue.*

In the sequel the following characerizations of density bases will be useful, its proof is left to the reader. We thank M. Hirayama for pointing us a mistake in a previous statement of Proposition B.9.

PROPOSITION B.9. *Each of the following are sufficient conditions for two systems $\{B_n(x)\}_x$ and $\{C_n(x)\}_x$ to be Vitali equivalent over a given σ-algebra $\mathcal{M}$:*

(1) *There exist $k \in \mathbb{Z}^+$ and $D > 0$ such that*

$$B_{n+k}(x) \subset C_n(x) \subset B_{n-k}(x) \qquad \text{with} \qquad \frac{m(B_{n+k}(x))}{m(B_n(x))} \geq D \qquad \text{for all } x \in M.$$

(2) *There exists $D > 0$ such that*

$$\frac{1}{D} \leq \frac{m(X \cap B_n(x))\, m(C_n(x))}{m(X \cap C_n(x))\, m(B_n(x))} \leq D \qquad \forall n \in \mathbb{Z}^+ \quad \forall X \in \mathcal{M}.$$

Proof of Lemma B.8.3. Observe that the system $\{B_{\sigma_n(x)}(x)\}$ is Vitali equivalent to Lebesgue since $\sigma_{n+1}(x)/\sigma_n(x) = \sigma(x)$ and $1/C < \sigma(x) < C$ for some $C > 1$. Observe also that, by Lemma B.6.1, if $y \in B^c_n(x)$ then $1/C < \sigma_n(x)/\sigma_n(y) < C$.

Now, if $z \in Q_n(x)$, then there are $y \in B^c_n(x)$ and $w \in \mathcal{W}^s_{\sigma_n(y)}(y)$ such that $z \in \mathcal{W}^u_{\sigma_n(x)}(w)$. So, we have that

$$d(z,y) \leq d(z,w) + d(w,y) + d(y,x) \leq \sigma(x) + \sigma(y) + \sigma(x) \leq (2 + C)\sigma(x).$$

Hence, for some fixed k, that do not depend on n, nor on x, $Q_n(x) \subset B_{\sigma_{n-k}(x)}(x)$.

To get the other inclusion, $B_{\sigma_n(x)}(x) \subset Q_{n-k}(x)$ for some fixed k, that does not depend on n or on x, we shall use the following lemma, which appeared in [12, Lemma 1.1] and is a consequence of the uniform continuity of the invariant bundles.

LEMMA B.9.1. *There are $\delta > 0$ small and $C > 0$ such that given four points, p_0, p_1, p_2, p_3 satisfying $p_1 \in W^c_{loc}(p_0)$, $p_2 \in \mathcal{W}^s_{loc}(p_1)$ and $p_3 \in \mathcal{W}^u_{loc}(p_2)$ then, if $d(p_0, p_3) < \delta$ we have that $d(p_i, p_{i+1}) \leq C d(p_0, p_3)$ for $i = 0, 1, 2$.*

Proof. The lemma follows by taking the line segments s_i from p_i to p_{i+1}, $i = 0, 1, 2$ and proving that the tangent vectors to this segments are uniformly linearly independent and this is true since this vectors are close to be in the corresponding bundles which form a uniform splitting. □

Let us finish the proof of Lemma B.8.3. Take $z \in B_{\sigma_n(x)}(x)$ and let us take the points, $w = \mathcal{W}^u_{loc}(z) \cap W^{sc}_{loc}(x)$ and $y = \mathcal{W}^s_{loc}(w) \cap W^c_{loc}(x)$. Then, applying Lemma B.9.1 to the points x, y, w, z in the position of p_0, p_1, p_2, p_3 respectively we get that $d(x, y), d(y, w)$ and $d(w, z)$ are less than some constant C times $d(x, z)$

and hence less that $C\sigma_n(x)$. Hence $y \in W^c_{C\sigma_n(x)}(x)$ so using Lemma B.6.1 we can take some $C' > C$ such that $C\sigma_n(x) < C'\sigma_n(y)$. Finally we get, taking k such that $C'\sigma_n < \sigma_{n-k}$ for ever $n \geq 0$, that $y \in B^c_{n-k}(x)$, $w \in \mathcal{W}^s_{\sigma_{n-k}(y)}(y)$ and $z \in \mathcal{W}^u_{\sigma_{n-k}(x)}(w)$, that is, $z \in Q_{n-k}(x)$.

The proof now follows from item (1) of Proposition B.9. □

Remark 4.2. *Observe that the definition of $Q_n(x)$ depends on the choice of $W^c_{loc}(x)$, but Lemma B.8.3 gives us that any choice will give equivalent basis and in fact equivalent to Lebesgue.*

Recall that a measurable set X is ***essentially** u**-saturated*** if there exists a measurable u-saturated set X_u (an ***essential** u**-saturate of*** X) such that $m(X \triangle X_u) = 0$.

Proposition B.10. *The system $\{J^{usc}_n(x)\}_{x\in M}$ is Vitali equivalent to $\{Q_n(x)\}_{x\in M}$ over essentially u-saturated sets.*

Proof. For measurable (small) sets X, let us denote by $m_u(X)$ and $m_{sc}(X)$ the induced Riemannian volume of X in $\mathcal{W}^u_{loc}$ and W^{sc}_{loc} respectively (the choice of W^{sc}_{loc} is fixed *a priori*). Since $\mathcal{W}^u$ is absolutely continuous, given any esentially u-saturated X, and any essential u-saturate X_u of X, we have

(1) $m(X_u \cap Q_n(x)) = \int_{X_u \cap J^{sc}_n(x)} m_u(\mathcal{W}^u_{\sigma_n(x)}(y))dm_{sc}(y)$,

(2) $m(X_u \cap J^{usc}_n(x)) = \int_{X_u \cap J^{sc}_n(x)} m_u(J^u_n(y))dm_{sc}(y)$.

Observe that there exists a constant $D > 1$ such that, for all $y \in J^{sc}_n(x)$,

$$\frac{1}{D} \leq \frac{m_u(J^u_n(y))}{m_u(J^u_n(x))} \leq D \tag{4.8}$$

(see Lemma 4.1. of [12]). Hence, we have,

$$\frac{1}{D^2}\frac{m_{sc}(X_u \cap J^{sc}_n(x))}{m_{sc}(J^{sc}_n(x))} \leq \frac{m(X_u \cap J^{usc}_n(x))}{m(J^{usc}_n(x))} \leq D^2\frac{m_{sc}(X_u \cap J^{sc}_n(x))}{m_{sc}(J^{sc}_n(x))}.$$

And also,

$$\frac{1}{D^2}\frac{m_{sc}(X_u \cap J^{sc}_n(x))}{m_{sc}(J^{sc}_n(x))} \leq \frac{m(X_u \cap Q_n(x))}{m(Q_n(x))} \leq D^2\frac{m_{sc}(X_u \cap J^{sc}_n(x))}{m_{sc}(J^{sc}_n(x))}.$$

So

$$\frac{1}{D^4}\frac{m(X \cap Q_n(x))}{m(Q_n(x))} \leq \frac{m(X \cap J^{usc}_n(x))}{m(J^{usc}_n(x))} \leq D^4\frac{m(X \cap Q_n(x))}{m(Q_n(x))}.$$

The claim follows now from Proposition B.9, part (2). □

Proposition B.11. *The system $\{J^{scu}_n(x)\}$ is Vitali equivalent to $\{J^{usc}_n(x)\}$ over all measurable sets.*

Proof. We shall find $l \in \mathbb{Z}^+$ and $D > 0$ such that

$$J^{scu}_{n+l}(x_0) \subset J^{usc}_n(x_0) \subset J^{scu}_{n-l}(x_0) \qquad \text{and} \qquad \frac{m(J^{usc}_{n+l}(x_0))}{m(J^{usc}_n(x_0))} \geq D$$

for all $x_0 \in M$ and $n > l$. The proof follows then from item (1) of Proposition B.9.

Let us consider $k_1 > k$, where k is the positive integer of Proposition B.8.2, satisfying $\min_{x\in M} \sigma_{-k_1}(x) > C^2$ where C is as in Lemma B.6.1. If $z \in J^{usc}_n(x_0)$, then $z \in U_n(y)$, with $y \in J^{sc}_n(x_0)$. By Lemma B.8.1 and the choice of k_1, we have $y \in B^c_{n-k_1}(x)$, with $x \in \mathcal{W}^s_{loc}(x_0)$. Applying Lemma B.8.2 to the holonomy map h^s going from $J^{cu}_{n-k_1}(x)$ to $W^{cu}_{loc}(x_0)$, we have $h^s(J^{cu}_{n-k_1}(x)) \subset J^{cu}_{n-2k_1}(x_0)$. Then, from the fact that the angles between distributions is bounded from below, we have that, for some $k_2 > k_1$, $z \in J^{cu}_{n-k_1}(x) \subset J^{scu}_{n-k_2}(x_0)$

The other inclusion is simpler, since, for $z \in J^{scu}_n(x_0)$, we have $z \in \mathcal{W}^s_{\sigma_n(y)}(y)$ with $y \in J^{cu}_n(x_0)$. But $W^{uc}_{loc}(z) \cap \mathcal{W}^s_{loc}(x_0) = \{x\}$, and hence directly from Lemma B.8.2 we have that z, belonging to $h^s(J^{cu}_n(x_0))$, is contained in $J^{cu}_{n-k_1}(x)$. Hence $z \in J^{usc}_{n-k_1}(x_0)$.

To finish the proof, let us see that $m(J^{usc}_{n+l}(x))/m(J^{usc}_n(x))$ is bounded from below for all $n > 0$ and $x \in M$. Proceeding as in Lemma B.10, we obtain that there is a constant $c > 0$ such that, for all $x \in M$ and $n > 0$

$$\frac{1}{c} \leq \frac{m(J^{usc}_n(x))}{m_u(J^u_n(x))m_s(\mathcal{W}^s_{\sigma_n(x)}(x))m_c(B^c_n(x))} \leq c.$$

It is easy to see that $m_s(\mathcal{W}^s_{\sigma_{n+l}(x)}(x))/m_s(\mathcal{W}^s_{\sigma_n(x)}(x))$ and $m_c(B^c_{n+k}(x))/m_c(B^c_n(x))$ are uniformly bounded. Now, using Lemma B.6.1, we have that

$$K^{-1}\frac{m_u(W^u_{\nu_n(x)}(f^n(x))}{\mathrm{Jac}(f^n)'(x)|_{E^u}} \leq m_u(J^u_n(x)) \leq K\frac{m_u(W^u_{\nu_n(x)}(f^n(x))}{\mathrm{Jac}(f^n)'(x)|_{E^u}}$$

for some uniform $K > 0$, so $m_u(J^u_{n+k}(x))/m_u(J^u_x(x))$ is uniformly bounded too. For a detailed proof of this last estimation see Lemma 4.4 of [12]. □

So let us prove Proposition B.3.

Proof of Proposition B.3. By Proposition B.11 we get that the system $\{J^{scu}_n(x)\}$ is Vitali equivalent to the system $\{J^{usc}_n(x)\}$ over all measurable sets. On the other hand, Proposition B.10 says that $\{J^{usc}_n(x)\}$ is equivalent to $\{Q_n(x)\}$ over u-saturated sets. Finally, by Lemma B.8.3 we know that $\{Q_n(x)\}$ is equivalent to Lebesgue over all measurable sets. hence we get that $\{J^{scu}_n(x)\}$ is Vitali equivalent to Lebesgue over u-saturated sets.

□

Finally we finish with the proof of Proposition B.2.

Proof of Proposition B.2. Let X_s be an essential s-saturate of X. And assume x is a J_n^{scu}- density point of X, hence of X_s. Calling $m_s(A)$ the induced Riemannian volume of A in $\mathcal{W}^s$, and $m_{cu}(A)$ the induced Riemannian volume of A in some (fixed a priori) W_{loc}^{cu} we have, due to the fact that X_s is s-saturated:

$$\frac{1}{K} \leq \frac{m(X_s \cap J_n^{scu}(x))}{\sigma_n(x) m_{cu}(X_s \cap J_n^{cu}(x))} \leq K$$

Now, due to Proposition B.8 we have

$$m_{cu}(h^s(X_s \cap J_{n+k}^{cu}(x))) \leq m_{cu}(X_s \cap J_n^{cu}(h^s(x))) \leq m_{cu}(h^s(X_s \cap J_{n-k}^{cu}(x)))$$

The proof follows from the fact that

$$\frac{1}{K} \leq \frac{m_{cu}(h^s(X))}{m_{cu}(X)} \leq K$$

for some uniform $K > 0$. □

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