ACCESSIBILITY AND ABUNDANCE OF ERGODICITY IN DIMENSION THREE: A SURVEY.

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Abstract. In [18] the authors proved the Pugh-Shub conjecture for partially hyperbolic diffeomorphisms with 1-dimensional center, i.e. stably ergodic diffeomorphisms are dense among the partially hyperbolic ones and, in subsequent results [20, 21], they obtained a more accurate description of this abundance of ergodicity in dimension three. This work is a survey type paper of this subject.

1. Introduction

The purpose of this survey is to present the state of the art in the study of the ergodicity of conservative partially hyperbolic diffeomorphisms in three dimensional manifolds. In fact, we shall mainly describe the results contained in [20, 21, 11]. The study of partial hyperbolicity has been one of the most active topics on dynamics over the last years and we do not pretend to describe all the related results, even for 3-manifolds. Some of the important themes excluded in this survey are entropy maximizing measures, absolute continuity of center foliations, co-cycles over partially hyperbolic systems, SRB-measures, dynamical coherence, classification, etc.

A diffeomorphism $f : M \to M$ of a closed smooth manifold M is partially hyperbolic if TM splits into three invariant bundles such that one of them is contracting, the other is expanding, and the third, called the center bundle, has an intermediate behavior, that is, not as contracting as the first, nor as expanding as the second (see the Subsection 2.3 for a precise definition). The first and second bundles are called strong bundles.

A central point in dynamics is to find conditions that guarantee ergodicity. In 1994, the pioneer work of Grayson, Pugh and Shub [9] suggested that partial hyperbolicity could be "essentially" a sufficient condition for ergodicity. Indeed, soon afterwards, Pugh and Shub conjectured that stable ergodicity (open sets of ergodic diffeomorphisms) is dense among partially hyperbolic systems. They proposed as an important tool the accessibility property (see also the previous work by Brin and Pesin [2]): f is accessible if any two points of M can be joined by a curve that is a finite union of arcs tangent to the strong bundles.

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Essential accessibility is the weaker property that any two measurable sets of positive measure can be joined by such a curve. In fact, accessibility will play a key role in this survey.

Pugh and Shub split their Conjecture into two sub-conjectures: (1) essential accessibility implies ergodicity, (2) the set of partially hyperbolic diffeomorphisms contains an open and dense set of accessible diffeomorphisms.

Many advances have been made since then in the ergodic theory of partially hyperbolic diffeomorphisms. In particular, there is a result by Burns and Wilkinson [4] proving that essential accessibility plus a bunching condition (trivially satisfied if the center bundle is one dimensional) implies ergodicity. There is also a result by the authors [18] obtaining the complete Pugh-Shub conjecture for one-dimensional center bundle. See [19] for a survey on the subject.

We have therefore that almost all partially hyperbolic diffeomorphisms with one dimensional bundle are ergodic. This means that the *non-ergodic* partially hyperbolic systems are very few. Can we describe them? Concretely,

QUESTION 1.1. *Which manifolds support a non-ergodic partially hyperbolic diffeomorphism? How do they look like?*

In this survey we give a description of what is known about this question for three dimensional manifolds. We study the sets of points that can be joined by paths everywhere tangent to the strong bundles (accessibility classes), and arrive, using tools of geometry of laminations and the topology of 3-manifolds, to the somewhat surprising conclusion that there are strong obstructions to the non-ergodicity of a partially hyperbolic diffeomorphism. See Theorems 1.4, 1.7 and 1.8.

This gave us enough evidence to conjecture the following:

CONJECTURE 1.2 ([20]). *The only orientable manifolds supporting non-ergodic partially hyperbolic diffeomorphisms in dimension 3 are the mapping tori of diffeomorphisms of surfaces which commute with Anosov diffeomorphisms.*

Specifically, they are (1) the mapping tori of Anosov diffeomorphisms of $\mathbb{T}^2$*, (2)* $\mathbb{T}^3$*, and (3) the mapping torus of* $-id$ *where* $id : \mathbb{T}^2 \to \mathbb{T}^2$ *is the identity map on the 2-torus.*

Indeed, we believe that for 3-manifolds, all partially hyperbolic diffeomorphisms are ergodic, unless the manifold is one of the listed above.

In the case that $M = \mathbb{T}^3$ we can be more specific and we also conjecture that:

CONJECTURE 1.3. *Let* $f : \mathbb{T}^3 \to \mathbb{T}^3$ *be a conservative partially hyperbolic diffeomorphism homotopic to a hyperbolic automorphism. . Then,* f *is ergodic.*

In [20] we proved Conjecture 1.2 when the fundamental group of the manifold is nilpotent:

Theorem 1.4. *All the conservative C^2 partially hyperbolic diffeomorphisms of a compact orientable 3-manifold with nilpotent fundamental group are ergodic, unless the manifold is $\mathbb{T}^3$.*

A paradigmatic example is the following. Let M be the mapping torus of $A_k : \mathbb{T}^2 \to \mathbb{T}^2$, where A_k is the automorphism given by the matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, k a non-zero integer. That is, M is the quotient of $\mathbb{T}^2 \times [0,1]$ by the relation $\sim$, where $(x,1) \sim (A_k x, 0)$. The manifold M has nilpotent fundamental group; in fact, it is a nilmanifold. Theorem 1.4 then implies that all conservative partially hyperbolic diffeomorphisms of M are ergodic.

Let us see that the above case, namely the case of nilmanifolds, is the only one where Theorem 1.4 is non-void. The Geometrization Conjecture, gives, after Perelman's work:

Theorem 1.5. *If M is a compact orientable manifold with nilpotent fundamental group, then either M is a nilmanifold or else it is finitely covered by $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{S}^1$.*

The second case mentioned in Theorem 1.5 is ruled out by a remarkable result by Burago and Ivanov:

Theorem 1.6 ([3]). *There are no partially hyperbolic diffeomorphisms in $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{S}^1$.*

The proofs of most of the theorems of this survey involve deep results of the geometry of codimension one foliations and the topology of 3-manifolds. In Section 2.1 we shall include, for completeness, the basic facts and definitions that we shall be using in this work. However, the interested reader is strongly encouraged to consult [5], [6], [12] and [13] for a well organized and complete introduction to the subject.

Let us explain a little bit our strategy. In the first place, it follows from the results in [4, 18] that accessibility implies ergodicity. So, our strategy will be to prove that all partially hyperbolic diffeomorphisms of compact 3-manifolds except the ones of the manifolds listed in Conjecture 1.2 satisfy the (essential) accessibility property.

In dimension 3, and in fact, whenever the center bundle is 1-dimensional, the non-open accessibility classes are codimension one immersed manifolds [18]; the set of non-open accessibility classes is a compact set *laminated* by the accessibility classes (see Section 2.1 for definitions). So, either f has the accessibility property or else there is a non-trivial lamination formed by non-open accessibility classes.

Let us first assume that the lamination is not a foliation (i.e. does not cover the whole manifold). Then in [20] it is showed that it either extends to a true foliation without compact leaves, or else it contains a leaf that is a periodic 2-torus with Anosov dynamics. In the first case, we have that the boundary leaves of the lamination contain a dense set of periodic points [18], and that their fundamental group injects in the fundamental group of the manifold. In the second case, let us call any embedded 2-torus admitting an Anosov dynamics extendable to the whole manifold, an *Anosov torus*. That is, $T \subset M$ is an Anosov torus if there exists a homeomorphism $h : M \to M$ such that $h|_T$ is homotopic to an Anosov diffeomorphism. In [21] we obtained that the manifold must be again one of the manifolds of Conjecture 1.2 if it has an Anosov torus.

Theorem 1.7. *A closed oriented irreducible 3-manifold admits an Anosov torus if and only if it is one of the following:*

(1) *the 3-torus*
(2) *the mapping torus of $-id$*
(3) *the mapping torus of a hyperbolic automorphism*

Let us recall that a 3-manifold is irreducible if any embedded 2-sphere bounds a ball. After the proof of the Poincaré conjecture this is the same of having trivial second fundamental group. Three dimensional manifolds supporting a partially hyperbolic diffeomorphism are always irreducible thanks to Burago and Ivanov results in [3]. Indeed, the existence of a Reebless foliation implies that the manifold is irreducible or it is $\mathbb{S}^2 \times \mathbb{S}^1$.

Secondly suppose that there are no open accessibility classes. Then, accessibility classes must foliate the whole manifold. Let us see that this foliation can not have compact leaves. Observe that any such compact leaf must be a 2-torus. So, we have three possibilities: (1) there is an Anosov torus, (2) the set of compact leaves forms a strict non-trivial lamination, (3) the manifold is foliated by 2-tori. The first case has just been ruled out. In the second case, we would have that the boundary leaves contain a dense set of periodic points, as stated above, and hence they would be Anosov tori again, which is impossible. Finally, in the third case, we conclude that the manifold is a fibration of tori over $\mathbb{S}^1$. This can only occur, in our setting, only if the manifold is the mapping torus of a diffeomorphism which commutes with an Anosov diffeomorphism as in Conjecture 1.2.

The following theorem is the first step in proving Conjectures 1.2 and 1.3. See definitions in Subsection 2.1:

Theorem 1.8. *Let $f : M \to M$ be a conservative partially hyperbolic diffeomorphism of an orientable 3-manifold M. Suppose that the bundles E^σ are also orientable, $\sigma = s, c, u$, and that f is not accessible. Then one of the following possibilities holds:*

(1) *M is the mapping torus of a diffeomorphism which commutes with an Anosov diffeomorphism as in Conjecture 1.2.*
(2) *there is an f-invariant lamination $\emptyset \neq \Gamma(f) \neq M$ tangent to $E^s \oplus E^u$ that trivially extends to a (not necessarily invariant) foliation without compact*

leaves of M. Moreover, the boundary leaves of $\Gamma(f)$ are periodic, have Anosov dynamics and dense periodic points.

(3) *there is a minimal invariant foliation tangent to $E^s \oplus E^u$.*

The assumption on the orientability of the bundles and M is not essential, in fact, it can be achieved by a finite covering. The proof of Theorem 1.8 appears at the end of Section 5.

We do not know of any example satisfying (2) in the theorem above. We have the following question.

QUESTION 1.9. *Let $f : N \to N$ be an Anosov diffeomorphism on a complete Riemannian manifold N. Is it true that if $\Omega(f) = N$ then N is compact?*

2. PRELIMINARIES

2.1. **Geometric preliminaries.** In this section we state several definitions and concepts that will be useful in the rest of this paper. From now on, M will be a compact connected Riemannian 3-manifold.

A *lamination* is a compact set $\Lambda \subset M$ that can be covered by open charts $U \subset \Lambda$ with a local product structure $\phi : U \to \mathbb{R}^n \times T$, where T is a locally compact subset of $\mathbb{R}^k$. On the overlaps $U_\alpha \cap U_\beta$, the transition functions $\phi_\beta \circ \phi_\alpha^{-1} : \mathbb{R}^n \times T \to \mathbb{R}^n \times T$ are homeomorphisms and take the form:

$$\phi_\beta \circ \phi_\alpha^{-1}(u, v) = (l_{\alpha\beta}(u, v), t_{\alpha\beta}(v)),$$

where $l_{\alpha\beta}$ are C^1 with respect to the u variable. No differentiability is required in the transverse direction T. The sets $\phi^{-1}(\mathbb{R}^n \times \{t\})$ are called *plaques*. Each point x of a lamination belongs to a maximal connected injectively immersed n-submanifold, called the *leaf* of x in L. The leaves are union of plaques. Observe that the leaves are C^1, but vary only continuously. The number n is the *dimension* of the lamination. If $n = \dim M - 1$, we say Λ is a *codimension-one* lamination. The set L is an *f-invariant lamination* if it is a lamination such that f takes leaves into leaves.

We call a lamination a *foliation* if $\Lambda = M$. In this case, we shall denote by $\mathcal{F}$ the set of leaves. In principle, we shall not assume any transverse differentiability. However, in case $l_{\alpha\beta}$ is C^r with respect to the v variable, we shall say that the foliation is C^r. Note that even purely C^0 codimension-one foliations admit a transverse 1-dimensional foliation (see Siebenmann [25],). In our case the existence of this 1-dimensional foliation is trivial thanks to the existence of the 1-dimensional center bundle E^c. These allows us to translate many local deformation arguments, usually given in the C^2 category, into the C^0 category as observed, for instance, by Solodov [26]. In particular, Theorems 2.1 and 2.3, which were originally formulated for C^2 foliations hold in the C^0 case. We shall say that a codimension-one foliation $\mathcal{F}$, is *transversely orientable* if the transverse

1-dimensional foliation mentioned above is orientable. An *invariant foliation* is a foliation that is an invariant lamination.

Let Λ be a codimension-one lamination that is not a foliation. A *complementary region* V is a component of $M \setminus \Lambda$. A *closed complementary region* $\hat{V}$ is the metric completion of a complementary region V with the path metric induced by the Riemannian metric, the distance between two points being the infimum of the lengths of paths in V connecting them. A closed complementary region is independent of the metric. Note that they are not necessarily compact. If Λ does not have compact leaves, then every closed complementary region decomposes into a compact *gut* piece and non-compact *interstitial regions* which are I-bundles over non-compact surfaces, and get thinner and thinner as they go away from the gut (see [13] or [8]). The interstitial regions meet the gut along annuli. The decomposition into interstitial regions and guts is unique up to isotopy. Moreover, one can take the interstitial regions as thin as one wishes.

A *boundary leaf* is a leaf corresponding to a component of ∂V, for V a closed complementary region. That is, a leaf is a non-boundary leaf if it is not contained in a closed complementary region.

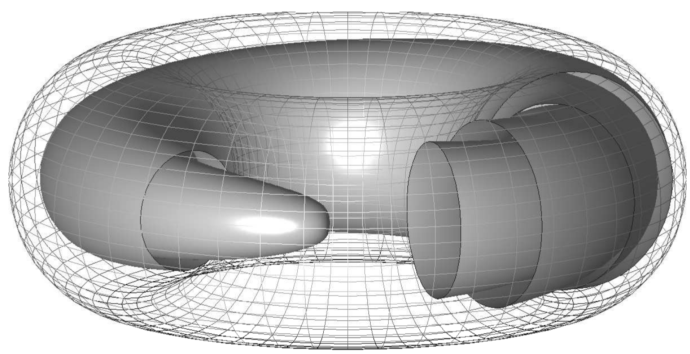

FIGURE 1. A Reeb component

The geometry of codimension-one foliations is deeply related to the topology of the manifold that supports them. The following subset of a foliation is important in their description. A *Reeb component* is a solid torus whose interior is foliated by planes transverse to the of core of the solid torus, such that each leaf limits on the boundary torus, which is also a leaf (see Figure 1). A foliation that has no Reeb components is called *Reebless*.

The following theorems show better the above mentioned relation:

THEOREM 2.1 (Novikov). *Let M be a compact orientable 3-manifold and $\mathcal{F}$ a transversely orientable codimension-one foliation. Then each of the following implies that $\mathcal{F}$ has a Reeb component:*

(1) *There is a closed, nullhomotopic transversal to $\mathcal{F}$*

(2) *There is a leaf L in $\mathcal{F}$ such that $\pi_1(L)$ does not inject in $\pi_1(M)$*

The statement of this theorem can be found, for instance, in [6] Theorems 9.1.3. 9.1.4., p.288. We shall also use the following theorem

THEOREM 2.2 (Haefliger). *Let Λ be a codimension one lamination in M. Then the set of points belonging to compact leaves is compact.*

This theorem was originally formulated for foliations [10]. However, it also holds for laminations, see for instance [13].

We have the following consequence of Novikov's Theorem about Reebless foliations. This theorem is stated in [24] as Corollary 2 on page 44.

THEOREM 2.3. *If M is a compact 3-manifold and $\mathcal{F}$ is a transversely orientable codimension-one Reebless foliation, then either $\mathcal{F}$ is the product foliation of $\mathbb{S}^2 \times \mathbb{S}^1$, or $\tilde{\mathcal{F}}$, the foliation induced by $\mathcal{F}$ on the universal cover $\tilde{M}$ of M, is a foliation by planes $\mathbb{R}^2$. In particular, if $M \neq \mathbb{S}^2 \times \mathbb{S}^1$ then M is irreducible.*

This theorem was originally stated for C^2 foliations, but it also holds for C^0 foliations, due to Siebenmann's theorem mentioned above.

2.2. **Topologic preliminaries.** Let M be a 3-dimensional manifold. A manifold M is *irreducible* if every 2-sphere $\mathbb{S}^2$ embedded in the manifold bounds a 3-ball. Recall that a 2-torus T embedded in M is an *Anosov torus* if there exists a diffeomorphism $f : M \to M$ such that $f(T) = T$ and the action induced by f on $\pi_1(T)$, that is, $f_{\#}|_T : \pi_1(T) \to \pi_1(T)$, is a hyperbolic automorphism. Equivalently, f restricted to T is isotopic to a hyperbolic automorphism.

We shall assume from now on, that M is an irreducible 3-manifold since this is the case for 3-manifolds supporting partially hyperbolic diffeomorphisms. In this subsection, we will focus on what is called the JSJ-decomposition of M (see below). That is, we will cut M along certain kind of tori, called incompressible, and will obtain certain 3-manifolds with boundary that are easier to handle, which are, respectively, Seifert manifolds, and atoroidal and acylindrical manifolds. Let us introduce these definitions first.

An orientable surface S embedded in M is *incompressible* if the homomorphism induced by the inclusion map $i_{\#} : \pi_1(S) \hookrightarrow \pi_1(M)$ is injective; or, equivalently, if there is no embedded disc $D^2 \subset M$ such that $D \cap S = \partial D$ and $\partial D \nsim 0$ in S (see, for instance, [12, Page 10]). We also require that $S \neq \mathbb{S}^2$.

A manifold with or without boundary is *Seifert*, if it admits a one dimensional foliation by closed curves, called a Seifert fibration. The boundary of an orientable Seifert manifold with boundary consists of finite union of tori. There are many examples of Seifert manifolds, for instance $\mathbb{S}^3$, T_1S where S is a surface, etc.

The other type of manifold obtained in the JSJ-decomposition is atoroidal and acylindrical manifolds. A 3-manifold with boundary N is *atoroidal* if every incompressible torus is *∂-parallel*, that is, isotopic to a subsurface of ∂N. A 3-manifold with boundary N is *acylindrical* if every incompressible annulus A that is *properly embedded*, i.e. $\partial A \subset \partial N$, is ∂-parallel, by an isotopy fixing ∂A.

As we mentioned before, a closed irreducible 3-manifold admits a natural decomposition into Seifert pieces on one side, and atoroidal and acylindrical components on the other:

THEOREM 2.4 (JSJ-decomposition [14], [15]). *If M is an irreducible closed orientable 3-manifold, then there exists a collection of disjoint incompressible tori $\mathcal{T}$ such that each component of $M \setminus \mathcal{T}$ is either Seifert, or atoroidal and acylindrical. Any minimal such collection is unique up to isotopy. This means, if $\mathcal{T}$ is a collection as described above, it contains a minimal sub-collection $m(\mathcal{T})$ satisfying the same claim. All collections $m(\mathcal{T})$ are isotopic.*

Any minimal family of incompressible tori as described above is called a *JSJ-decomposition* of M. When it is clear from the context we shall also call JSJ-decomposition the set of pieces obtained by cutting the manifold along these tori. Note that if M is either atoroidal or Seifert, then $\mathcal{T} = \emptyset$.

2.3. **Dynamic preliminaries.** Throughout this paper we shall work with a *partially hyperbolic diffeomorphism* f, that is, a diffeomorphism admitting a non-trivial Tf-invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$, such that all unit vectors $v^\sigma \in E^\sigma_x$ ($\sigma = s, c, u$) with $x \in M$ verify:

$$\|T_x f v^s\| < \|T_x f v^c\| < \|T_x f v^u\|$$

for some suitable Riemannian metric. f also must satisfy that $\|Tf|_{E^s}\| < 1$ and $\|Tf^{-1}|_{E^u}\| < 1$. We shall say that a partially hyperbolic diffeomorphism f that satisfies

$$\|T_x f v^s\| < \|T_y f v^c\| < \|T_z f v^u\| \; \forall x, y, z \in M$$

is *absolutely partially hyperbolic*.

We shall also assume that f is *conservative*, i.e. it preserves Lebesgue measure associated to a smooth volume form.

It is a known fact that there are foliations $\mathcal{W}^\sigma$ tangent to the distributions E^σ for $\sigma = s, u$ (see for instance [2]). The leaf of $\mathcal{W}^\sigma$ containing x will be called $W^\sigma(x)$, for $\sigma = s, u$. The connected component of x in the intersection of $W^s(x)$ with a small ε-ball centered at x is the *ε-local stable manifold of* x, and is denoted by $W^s_\varepsilon(x)$.

In general it is not true that there is a foliation tangent to E^c. It is false even in case $\dim E^c = 1$ (see [22]). However, in Proposition 3.4 of [1] it is shown that if $\dim E^c = 1$, then f is *weakly dynamically coherent*. This means, for each $x \in M$ there are complete immersed C^1 manifolds which contain x and are everywhere tangent to E^c, E^{cs} and E^{cu}, respectively. We will call a *center curve* any curve which is everywhere tangent to E^c. Moreover, we will use the following fact:

Proposition 2.5 ([1]). *If γ is a center curve through x, then*

$$W^s_\varepsilon(\gamma)=\bigcup_{y\in\gamma}W^s_\varepsilon(y) \qquad \text{and} \qquad W^u_\varepsilon(\gamma)=\bigcup_{y\in\gamma}W^u_\varepsilon(y)$$

are C^1 immersed manifolds everywhere tangent to $E^s\oplus E^c$ and $E^c\oplus E^u$ respectively.

We shall say that a set X is *s-saturated* or *u-saturated* if it is a union of leaves of the strong foliations $\mathcal{W}^s$ or $\mathcal{W}^u$ respectively. We also say that X is su-saturated if it is both s- and u-saturated. The *accessibility class* $AC(x)$ of the point $x\in M$ is the minimal su-saturated set containing x. Note that the accessibility classes form a partition of M. If there is some $x\in M$ whose accessibility class is M, then the diffeomorphism f is said to have the *accessibility property*. This is equivalent to say that any two points of M can be joined by a path which is piecewise tangent to E^s or to E^u. A diffeomorphism is said to be *essentially accessible* if any su-saturated set has full or null measure.

The theorem below relates accessibility with ergodicity. In fact it is proven in a more general setting, but we shall use the following formulation:

Theorem 2.6 ([4],[18]). *If f is a C^2 conservative partially hyperbolic diffeomorphism with the (essential) accessibility property and* $\dim E^c=1$*, then f is ergodic.*

In [20] it is proved that there are manifolds whose topology implies the accessibility property holds for all partially hyperbolic diffeomorphisms. In these manifolds, all partially hyperbolic diffeomorphisms are ergodic.

Sometimes we will focus on the openness of the accessibility classes. Note that the accessibility classes form a partition of M. If all of them are open then, in fact, f has the accessibility property. We will call $U(f)=\{x\in M; AC(x)$ is open$\}$ and $\Gamma(f)=M\setminus U(f)$. Note that f has the accessibility property if and only if $\Gamma(f)=\emptyset$. We have the following property of non-open accessibility classes:

Proposition 2.7 ([18]). *The set $\Gamma(f)$ is a codimension-one lamination, having the accessibility classes as leaves.*

In fact, any compact su-saturated subset of $\Gamma(f)$ is a lamination.

The above proposition is Proposition A.3. of [18]. The fact that the leaves of $\Gamma(f)$ are C^1 may be found in [7]. The following proposition is Proposition A.5 of [18]:

Proposition 2.8 ([18]). *If Λ is an invariant sub-lamination of $\Gamma(f)$, then each boundary leaf of Λ is periodic and the periodic points are dense in it (with the induced topology).*

Moreover, the stable and unstable manifolds of each periodic point are dense in each plaque of a boundary leaf of Λ.

Observe that the proof of Proposition A.5 of [18] shows in fact that periodic points are dense in the accessibility classes of the boundary leaves of V endowed with its intrinsic topology. In other words, periodic points are dense in each plaque of the boundary leaves of V.

We shall also use the following theorem by Brin, Burago and Ivanov, whose proof is in [1], after Proposition 2.1.

Theorem 2.9 ([1]). *If $f : M^3 \to M^3$ is a partially hyperbolic diffeomorphism, and there is an open set V foliated by center-unstable leaves, then there cannot be a closed center-unstable leaf bounding a solid torus in V.*

3. Anosov tori

In this section we will say a few words about the proof of Theorem 1.7. The idea in its proof is that, given an Anosov torus T, we can "place" T so that either T belongs to the family $\mathcal{T}$ given by the JSJ-decomposition (Theorem 2.4), or else T is in a Seifert component, and it is either transverse to all fibers, or it is union of fibers of this Seifert component. See Proposition 3.3.

It is important to note the following property of Anosov tori:

Theorem 3.1 ([20]). *Anosov tori are incompressible.*

An Anosov torus in an atoroidal component will then be ∂-parallel to a component of its boundary. In this case, we can assume $T \in \mathcal{T}$. On the other hand, the Theorem of Waldhausen below, guarantees that we can always place an incompressible torus in a Seifert manifold in a "standard" form; namely, the following: a surface is *horizontal* in a Seifert manifold if it is transverse to all fibers, and *vertical* if it is union of fibers:

Theorem 3.2 (Waldhausen [27]). *Let M be a compact connected Seifert manifold, with or without boundary. Then any incompressible surface can be isotoped to be horizontal or vertical.*

The architecture of the proof of Theorem 1.7 is contained in the following proposition.

Proposition 3.3. *Let T be an Anosov torus of a closed irreducible orientable manifold M. Then, there exists a diffeomorphism $f : M \to M$ and a JSJ-decomposition $\mathcal{T}$ such that*

(1) *$f|T$ is a hyperbolic toral automorphism,*
(2) *$f(\mathcal{T}) = \mathcal{T}$, and*
(3) *one of the following holds*
    (a) $T \in \mathcal{T}$
    (b) *T is a vertical torus in a Seifert component of $M \setminus \mathcal{T}$, and T is not ∂-parallel in this component.*

(c) *M is a Seifert manifold ($\mathcal{T} = \emptyset$), and T is a horizontal torus,*

The proposition above allows us to split the proof of Theorem 1.7 into cases. Note that case (3b) includes the case in which M is a Seifert manifold and T is a vertical torus.

In the case that T is a vertical torus in a Seifert component we can cut this component along T. Then we can suppose that T is in the boundary. We take profit of the fact that in most manifolds the Seifert fibration is unique up to isotopy. Since the dynamics restricted to T is Anosov we have that the manifold has more than one Seifert fibration. This lead us to show that this Seifert component must be $\mathbb{T}^2 \times [0, 1]$. This gives that the whole manifold must be one of the manifolds of Theorem 1.7.

If T is horizontal torus then the manifold M is Seifert and T intersects all the fibers. This is discarded in a case by case study thanks to the fact that the Seifert manifolds having horizontal torus a finite.

The last and more difficult case is when T is part of the JSJ-decomposition but it is not the boundary of a Seifert component. The proof in this case is complicated but a very rough idea is to take a properly embedded surface S with an essential circle of T in its boundary. Taking a large iterate $f^n(S)$ and considering $S \cap f^n(S)$, it is possible to construct a non-parallel incompressible cylinder as a union of a band in S and a band in $f^n(S)$. This leads to contradiction because the component is not Seifert and then, it is acylindrical.

4. The su-lamination $\Gamma(f)$

Let f be a partially hyperbolic diffeomorphism of a compact 3-manifold M. From Subsection 2.3 it follows that we have three possibilities: (1) f has the accessibility property, (2) the set of non-open accessibility classes is a strict lamination, $\emptyset \subsetneq \Gamma(f) \subsetneq M$ or (3) the set of non-open accessibility classes foliates M: $\Gamma(f) = M$.

Now, we shall distinguish two possible cases in situations (2) and (3):

(a) the lamination $\Gamma(f)$ does not contain compact leaves
(b) the lamination $\Gamma(f)$ contains compact leaves

In this section we deal with the case (2a). In fact, for our purposes it will be sufficient to assume that there exists an f-invariant sub-lamination Λ of $\Gamma(f)$ without compact leaves. Section 5 treats the cases (2b) and (3b). Section 6 treats the case (3a).

In this section, we will prove that the complement of Λ consists of I-bundles. To this end, we shall assume that the bundles E^σ ($\sigma = s, c, u$) and the manifold M are orientable (we can achieve this by considering a finite covering).

THEOREM 4.1 ([20], Theorem 4.1). *If $\emptyset \subsetneq \Lambda \subset \Gamma(f)$ is an orientable and transversely orientable f-invariant sub-lamination without compact leaves such that $\Lambda \neq M$, then all closed complementary regions of Λ are I-bundles.*

Theorem 4.1 was proved by showing:

PROPOSITION 4.2. *Let $\Lambda \subset \Gamma(f)$ be a nonempty f-invariant sub-lamination without compact leaves. Then E^c is uniquely integrable in the closed complementary regions of Λ.*

The proof of this proposition is rather technical. The interested reader may found a proof in [20].

Let us consider $\hat{V}$ a closed complementary region of Λ, and call $\mathcal{I}(V)$ the union of all interstitial regions of V and $\mathcal{G}(V)$ the gut of $\hat{V}$ (see Subsection 2.1), so that

$$\hat{V} = \mathcal{I}(V) \cup \mathcal{G}(V).$$

The following statement is rather standard:

LEMMA 4.3. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism. If U is an open invariant set such that $U \subset \Omega(f)$, then the closure of U is su-saturated.*

Let us observe that if $\hat{V}$ is connected then there are only two boundary leaves of $\hat{V}$. Indeed, as we mentioned before periodic points are dense in boundary leaves. This fact jointly with the local product structure imply, using standard arguments, that the stable and unstable leaves of periodic points are dense too. Take a periodic point p in a boundary leaf and in the intersticial region. There are center curves joining the points in the local stable manifold of p with other boundary curve L_1 of $\hat{V}$ (the same property holds for the local unstable manifold). Invariance of the stable manifold of p and boundary leaves give that the center curve of any point of the stable manifold joins the boundary leaf L_0 containing p with L_1. Denseness of the stable and unstable manifolds of p implies that the complement of the set of points such that their center manifold join L_0 with L_1 is totally disconnected. Then, it is not difficult to see that L_0 and L_1 are the unique boundary leaves of $\hat{V}$.

Also, since periodic points are dense in the boundary leaves due to Proposition 2.8, there is an iterate of f that fixes all connected components of $\hat{V}$, so we will assume when proving Theorem 4.1 that $\hat{V}$ is connected and has two boundary leaves L_0 and L_1.

Proof of Theorem 4.1. We will present a sketch of a different approach to a proof than the one given in [20]. The strategy will be to show that all center leaves in $\hat{V}$ meet both L_0 and L_1. Let p be a periodic point in $L_0 \cap \mathcal{I}(V)$. As we mentioned before its center leaf meets L_1, and the same happens for all points in its stable and unstable manifolds. Now stable and unstable manifolds of a periodic point are dense in each plaque of L_0 (Proposition 2.8). So the set of points in L_0 whose center leaf does not reach L_1 is contained in a totally disconnected set.

Let us suppose that x_0 is a point in L_0 whose center leaf does not reach L_1. Then, since center curves of points of the intersticial region clearly reach the boundary, $W^c(x_0)$ is contained in $\mathcal{G}(V)$. Take a small rectangle R in L_0 around x_0 formed by arcs of stable and unstable manifolds of a periodic point. Moreover, we can assume that the center curves of the points of R_0 reach L_1. Of course, the image is another rectangle R_1 formed by stable and unstable arcs. Then, the center arcs of the points of R_0 and the interiors of R_0 and R_1 form a 2-sphere S. Since Rosenberg's theorem [23] remains valid in this setting and $\hat{V}$ is foliated by $\mathcal{W}^{cs}$ that is Reebless and transverse to the boundary, we have that $\hat{V}$ is irreducible. Then, S bounds a ball B. Now, since $W^c(x_0)$ does not reach L_1 and is contained in B, it accumulates in B but Novikov's Theorem implies the existence of a Reeb component, a contradiction. □

Theorem 4.1 implies that any non trivial invariant sub-lamination $\Lambda \subset \Gamma(f)$ without compact leaves can be extended to a foliation of M without compact leaves. Indeed, any complementary region V is an I-bundle, and hence it is diffeomorphic to the product of a boundary leaf times the open interval: $L_0 \times (0,1)$. The foliation $F_t = L_0 \times \{t\}$ induces a foliation of V.

This has the following consequence in case the fundamental group of M is nilpotent:

PROPOSITION 4.4. *If M is a compact 3-manifold with nilpotent fundamental group, and $\emptyset \subsetneq \Lambda \subsetneq M$, is an invariant sub-lamination of $\Gamma(f)$, then there exists a leaf of Λ that is a periodic 2-torus with Anosov dynamics.*

Proof. If Λ has a compact leaf, let us consider the set Λ_c of all compact leaves of Λ. Λ_c is in fact an invariant sub-lamination, due to Theorem 2.2. Hence Proposition 2.8 implies that the boundary leaves of Λ_c are periodic 2-tori with Anosov dynamics, and we obtain the claim.

If, on the contrary, Λ does not have compact leaves, then due to Theorem 4.1 above, we can extend Λ to a foliation $\mathcal{F}$ of M without compact leaves. In particular, $\mathcal{F}$ is a Reebless foliation. Item (2) of Theorem 2.1 implies that for all boundary leaves L of Λ, $\pi_1(L)$ injects in $\pi_1(M)$, and is therefore nilpotent.

Now, this implies that the boundary leaves can only be planes or cylinders. Theorem 2.8 implies that stable and unstable leaves of periodic points are dense in those leaves, which is impossible for the case of the plane or the cylinder. Therefore, Λ must contain a compact leaf, and due to what was shown above, it must contain a periodic 2-torus with Anosov dynamics. □

In fact, Theorem 1.7 implies that periodic 2-tori with Anosov dynamics are not possible in 3-manifolds with nilpotent fundamental group, unless the manifold is

$\mathbb{T}^3$. Hence the hypotheses of Proposition 4.4 are not fulfilled, unless the manifold is $\mathbb{T}^3$. This will eliminate case (2) mentioned at the beginning of this section.

5. The trichotomy of Theorem 1.8

In this section we will prove Theorem 1.8. This theorem and the results in this section are valid for any 3-manifold M, and do not require that its fundamental group be nilpotent. Moreover, Theorem 3.1 does not even require the existence of a partially hyperbolic diffeomorphism.

Let T be an embedded 2-torus in M. We shall call T an ***Anosov torus*** if there exists a homeomorphism $g : M \to M$ such that T is g-invariant, and $g|_T$ is homotopic to an Anosov diffeomorphism.

Also, let S be a two-sided embedded closed surface of M^3 other than the sphere. S is ***incompressible*** if and only if the homomorphism induced by the inclusion map $i_\# : \pi_1(S) \hookrightarrow \pi_1(M)$ is injective; or, equivalently, after the Loop Theorem, if there is no embedded disc $D^2 \subset M$ such that $D \cap S = \partial D$ and $\partial D \nsim 0$ in S (see, for instance, [12]).

Recall that Theorem 3.1 says that Anosov tori are incompressible. We insist that this theorem is general, and does not depend on the existence of a partially hyperbolic dynamics in the manifold.

We also need the following fact about codimension one laminations.

Theorem 5.1. *Let $\mathcal{F}$ be a codimension one C^0-foliation without compact leaves of a three dimensional compact manifold M. Then, $\mathcal{F}$ has a finite number of minimal sets.*

We are now in position to prove Theorem 1.8 of Page 4:

Proof of Theorem 1.8. If $\Gamma(f) = M$ then there are no Reeb components. Indeed, since f is conservative, if there were a Reeb component, then its boundary torus should be periodic. We get a contradiction from Theorem **??**. This gives case (3) except the minimality.

Let us assume that $\Gamma(f) \neq M$. If $\Gamma(f)$ contains a compact leaf then the set of compact leaves is a sub-lamination Λ of $\Gamma(f)$ by Theorem 2.2. Proposition 2.8 implies that the boundary leaves of Λ are Anosov tori, and we obtain case (1) as a consequence of Theorem 1.7.

If $\Gamma(f) \neq M$ and contains no compact leaves, then Theorem 4.1 and Proposition 2.8 give us case (2).

Finally we show minimality in case (3). On the one hand, if $\Gamma(f) = M$ and has a compact leaf we have two possibilities: either all leaves are compact or not. If not then, the previous argument implies the existence of an Anosov torus and we are in case (1). If all leaves are compact, as we mentioned before, the manifold is a torus bundle and the hyperbolic dynamics on fibers implies that we are again in case (1). On the other hand, if $\Gamma(f)$ has no compact leaves and has a minimal sub-lamination $\mathcal{L}$, we have that $\mathcal{L}$ is periodic (recall that minimal sub-laminations of a codimension one foliation are finite, Theorem 5.1). Then, we are again in case (2). □

6. Nilmanifolds

Let $f : M \to M$ be a conservative partially hyperbolic diffeomorphism of a compact orientable three dimensional nilmanifold $M \neq \mathbb{T}^3$. As consequence of Proposition 4.4 and Theorems 1.7 and 1.7 we have that $E^s \oplus E^u$ integrates to a minimal foliation $\mathcal{F}^{su}$ if f does not have the accessibility property. Indeed the only possibilities in the trichotomy of Theorem 1.8 are (2) and (3) and Proposition 4.4 says that there is an Anosov torus if we are in case (2). But this last case is impossible for a nilmanifold $M \neq \mathbb{T}^3$. In this section we shall give some arguments showing that the existence of such a foliation leads us to a contradiction. In [20] the reader can find a different proof of the same fact. Without loss of generality we may assume, by taking a double covering if necessary, that $\mathcal{F}^{su}$ is transversely orientable. Observe that the double covering of a nilmanifold is again a nilmanifold.

The first step is that Parwani [16] proved (following Burago-Ivanov [3] arguments) that the action induced by f in the first homology group of M is hyperbolic. By duality the same is true for the first cohomology group.

The second step is given by Plante results in [17] (see also [13]). Since $\mathcal{F}^{su}$ is a minimal foliation of a manifold whose fundamental group has non-exponential growth there exists a transverse holonomy invariant measure μ of full support. This measure is unique up to multiplication by a constant and represents an element of the first cohomology group of M. The action of f leaves $\mathcal{F}^{su}$ invariant and induces a new transverse measure ν, an image of former one. The uniqueness implies that $\nu = \lambda\mu$ for some $\lambda > 0$. Since of the action of f on $H^1(M)$ is hyperbolic, then $\lambda \neq 1$. Suppose that $\lambda > 1$ (if the contrary is true take f^{-1}).

The third step is to observe that $\lambda > 1$ implies that f is expanding the μ measure of center curves. Since μ has full support and the $su-$bundle is hyperbolic we would obtain that f is conjugated to Anosov leading to contradiction with the fact that $M \neq \mathbb{T}^3$.

7. $M = \mathbb{T}^3$

In this section we present the results announced by Hammerlindl and Ures on Conjecture 1.3, that the nonexistence of nonergodic partially hyperbolic diffeomorphisms homotopic to Anosov in dimension 3. They are able to prove the following result.

Theorem 7.1 ([11]). *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a $C^{1+\alpha}$ conservative partially hyperbolic diffeomorphism homotopic to a hyperbolic automorphism A. Suppose that f is not ergodic. Then,*

(1) *$E^s \times E^u$ integrates to a minimal foliation.*
(2) *f is topologically conjugated to A and the conjugacy sends strong leaves of f into the corresponding strong leaves of A.*
(3) *The center Lyapunov exponent is* 0 *a.e.*

We remark that it is not known if there exists a diffeomorphism satisfying the conditions of the theorem above.

Now, in order to prove Conjecture 1.3 we have two possibilities: either we prove that a diffeomorphism satisfying the conditions of Theorem 7.1 is ergodic or we prove that such a diffeomorphism cannot exist. Hammerlindl and Ures announced that if f is C^2 and the center stable and center unstable leaves of a periodic point are C^2 then, f is ergodic.

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